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DIPLOMOVÁ PRÁCE

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Kvalitativní vlastnosti řešení rovnic popisujících časově proměnná proudění nestlačitelných chemicky reagujících tekutin

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Abstrakt: V práci studujeme vlastnosti řešení systému rovnic, který popisuje časově proměnné proudění nestlačitelné kapaliny s příměsí, jejíž viskozita závisí na koncentraci příměsi. Uvažujeme dvojrozměrný případ a pro viskozitu, která je zdola omezená kladnou konstantou a závisí na koncentraci lipschitzovsky, dokazujeme, že druhé prostorové derivace rychlosti a koncentrace leží v prostoru $L^2(L^2)$ a první prostorové derivace v prostoru $L^\infty(L^2)$. Tento globální výsledek o regularitě pak využijeme k důkazu jednoznačnosti tohoto řešení ve třídě slabých řešení.
Klíčová slova: parciální diferenciální rovnice, mechanika tekutin, regularita

Title: Qualitative properties of solutions of evolutionary equations describing flows of incompressible chemically reacting fluids
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Abstract: We study properties of solutions of generalized Navier-Stokes equations describing unsteady flows of an incompressible fluid with impurity, whereas the viscosity of the fluid depends on concentration of the impurity. We assume that the viscosity is function of the concentration, which is bounded and Lipschitz continuous. In two dimensional case we prove that the second spatial derivatives of the velocity and concentration belong to the space $L^2(L^2)$ and first spatial derivatives are in $L^\infty(L^2)$. As the application of this global regularity result we prove uniqueness of the regular solution within the class of weak solutions.
Keywords: partial differential equations, fluid mechanics, regularity

Chapter 1

Introduction

1.1 Formulation of the problem

Fluid mechanics is an old branch of classical physics. Although fluids were studied from mechanical point of view since the period of formation of building blocks of mechanics by Newton, the equations which describe flows of the Newtonian fluids (such as air or water under normal conditions) were discovered two hundred years after Newton by Navier, Poisson, Saint-Venant and Stokes. Until today it is believed until today that Navier-Stokes equations describe such fluids with sufficient accuracy. There exist fluids (such as starch suspensions, shampoo or blood) which exhibits phenomena like shear thickening, shear thinning and stress relaxation which cannot be described by Navier - Stokes equation. For these fluids more complicated constitutive relations have to be considered.

In this thesis we study mathematical properties of fluids with more complicated structure then simple Navier-Stokes fluid. We consider model from the article [1], which describes behavior of fluid, whose viscosity depends on the concentration of some impurity, or fluid which reacts chemically and its viscosity is dependent on the chemical state of the fluid. Both these interpretations are possible provided that the variables in the equations are interpreted in the appropriate way. Apart from velocity and pressure there is also quantity, which we will call concentration. In the case of fluid with impurity the concentration can be interpreted as ratio of the density of impurity and the density of whole fluid. In the case of chemically reacting fluid the concentration can not be interpreted in such simple way, but has to be viewed as quantity, which describes chemical state of the fluid.

We will study behavior of the fluid flowing inside a bounded domain $\Omega \subset \mathbf{R}^d$ over time interval $[0, T]$. Here d is dimension of the space and Ω is supposed to have sufficiently smooth boundary $\partial\Omega$, its minimal smoothness will be specified later. In the main part of the thesis we will consider only the case $d = 2$. We will describe the fluid motion by its velocity $\boldsymbol{v} \times [0, T] : \Omega \to \mathbf{R}^d$, pressure $p : \Omega \times [0, T] \to \mathbf{R}$, concentration $c : \Omega \times [0, T] \to \mathbf{R}$, viscous part of the stress tensor $\mathbf{S} : \Omega \times [0, T] \to \mathbf{R}^{d\times d}_{sym}$ and concentration flux $\boldsymbol{q}_c : \Omega \times [0, T] \to \mathbf{R}^d$. We will consider only incompressible fluid with constant density, therefore

$$\operatorname{div} \boldsymbol{v} = 0. \tag{1.1}$$

The flow is governed by balance of linear momentum,

$$\boldsymbol{v}_{,t} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbf{S} = -\nabla p + \boldsymbol{f} \tag{1.2}$$

and by convection diffusion equation for concentration,

$$c_{,t} + \operatorname{div}(\boldsymbol{v}c) + \operatorname{div} \boldsymbol{q}_c = 0. \tag{1.3}$$

Balance of angular momentum is satisfied automatically since $\mathbf{S}$ is supposed to be symmetric. We will further suppose that viscous part of stress tensor $\mathbf{S}$ is function of concentration and the symmetric part of the velocity gradient $\mathbf{D} = \mathbf{D}(\boldsymbol{v}) = \frac{1}{2}\left(\nabla \boldsymbol{v} + [\nabla \boldsymbol{v}]^T\right)$,

$$\mathbf{S} = \mathbf{S}(c, \mathbf{D}) = \nu(c, |\mathbf{D}|^2)\mathbf{D},$$

and flux of the concentration $\boldsymbol{q}_c$ depends on the shear rate and linearly on concentration gradient :

$$\boldsymbol{q}_c = \boldsymbol{q}_c(c, \nabla c, \mathbf{D}) = \mathbf{K}(c, |\mathbf{D}|^2)\nabla c.$$

Further we have to add boundary conditions for concentration and velocity. We will consider impermeability of the wall of Ω, therefore

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{ on } \Gamma, \tag{1.4}$$

where $\boldsymbol{n}$ denotes unit outward normal to the boundary of Ω and $\Gamma = \partial\Omega \times [0, T]$. We will consider the Navier partial slip boundary condition for the velocity

$$\alpha \boldsymbol{v}_\tau + (\mathbf{S}\boldsymbol{n})_\tau = 0 \text{ on } \Gamma, \tag{1.5}$$

where α is positive constant and index τ denotes tangent component to the boundary of Ω. For the concentration we will consider zero flux of concentration through $\Gamma_N \subset \Omega \times [0, T]$ and Dirichlet boundary condition for concentration on $\Gamma_D \subset \Gamma$

$$\begin{aligned} \boldsymbol{q}_c \cdot \boldsymbol{n} &= 0 \quad \text{ on } \Gamma_N, \\ c &= c_b \quad \text{ on } \Gamma_D, \end{aligned} \tag{1.6}$$

where c_b is given function on Γ_D, which satisfies $0 \le c_b \le 1$.

1.2 Derivation of the governing equation from the mixture theory

To derive the equations we use mixture theory as presented in [7]. We will assume that our fluid has two components, each described by density ρ^α, velocity $\boldsymbol{v}^\alpha$ and concentration c^α ($\alpha \in \{1, 2\}$). We set

$$\begin{aligned} \rho &= \rho^1 + \rho^2, \\ \boldsymbol{v} &= \frac{1}{\rho}\left(\rho^1 \boldsymbol{v}^1 + \rho^2 \boldsymbol{v}^2\right), \end{aligned}$$

density and velocity of whole mixture. In whole derivation we will assume that ρ is constant. This requirement is met when density of the mixture is not dependent on concentration of the mixture and is satisfied approximately when the differences of concentration in the fluid are small enough not to influence the density significantly. This assumption is fulfilled well for example for synovial fluids.

As a first step we derive balance of mass for the whole mixture. We will start from balances of momentum for individual components :

$$\frac{\partial \rho^\alpha}{\partial t} + \operatorname{div}\left(\rho^\alpha \boldsymbol{v}^\alpha\right) = 0. \tag{1.7}$$

We sum (1.7) over α, use definitions of $\boldsymbol{v}$ and ρ, and obtain

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho \boldsymbol{v}\right) = 0.$$

Since we assumed ρ to be constant this equation reduces to incompressibility condition,

$$\operatorname{div} \boldsymbol{v} = 0.$$

Now we derive the equation for concentration. We set $\boldsymbol{v}_d^\alpha = \boldsymbol{v} - \boldsymbol{v}^\alpha$ and calculate material derivative of concentrations $c^\alpha = \frac{\rho^\alpha}{\rho}$. Since ρ is assumed to be constant the calculation takes the following form :

$$\rho \dot{c}^\alpha = \rho \frac{\partial c^\alpha}{\partial t} + \rho \nabla c^\alpha \cdot \boldsymbol{v} = \frac{\partial \rho^\alpha}{\partial t} + \nabla \rho^\alpha \cdot \boldsymbol{v}.$$

We use the equations balancing the mass of individual components and the incompressibility constraint and observe that

$$-\operatorname{div}\left(\rho^\alpha \boldsymbol{v}^\alpha\right) + \operatorname{div}\left(\rho^\alpha \boldsymbol{v}\right) = \operatorname{div}(\rho^\alpha \boldsymbol{v}_d^\alpha).$$

Consequently,

$$\rho \frac{\partial c^\alpha}{\partial t} + \rho \nabla c^\alpha \cdot \boldsymbol{v} = \operatorname{div}(\rho^\alpha \boldsymbol{v}_d^\alpha). \tag{1.8}$$

Dividing 1.8 by constant density ρ gives us

$$\frac{\partial c^\alpha}{\partial t} + \nabla c^\alpha \cdot \boldsymbol{v} = \operatorname{div}(c^\alpha \boldsymbol{v}_d^\alpha).$$

We assume that $c^\alpha \boldsymbol{v}_d^\alpha$ being nonzero is caused by diffusion of components of mixture. We assume that rate of diffusion (concentration flux) depends on ∇c linearly with matrix $\mathbf{K}$(Fick law), therefore $c^\alpha \boldsymbol{v}_d^\alpha = \mathbf{K} \nabla c^\alpha$. We get the equation

$$\frac{\partial c^\alpha}{\partial t} + \nabla c^\alpha \cdot \boldsymbol{v} = \operatorname{div}(\mathbf{K} \nabla c^\alpha) \tag{1.9}$$

Now we derive balance of linear momentum for whole mixture. We start from balances of momentum of individual components.

$$\frac{\partial \rho^\alpha \boldsymbol{v}^\alpha}{\partial t} + \operatorname{div}\left(\rho^\alpha \boldsymbol{v}^\alpha \otimes \boldsymbol{v}^\alpha\right) = \operatorname{div} \mathbf{T}^\alpha + \boldsymbol{I}^\alpha + \rho^\alpha \boldsymbol{f} \tag{1.10}$$

Where $\mathbf{T}^\alpha$ is the stress tensor of component α and $\boldsymbol{I}^\alpha$ is the force exerted by the other component on component α. We sum the balances of momentum of the individual components. Since Newton's third law holds, $\boldsymbol{I}^1 + \boldsymbol{I}^2 = 0$, we have

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \operatorname{div}\left(\rho^1 \boldsymbol{v}^1 \otimes \boldsymbol{v}^1 + \rho^2 \boldsymbol{v}^2 \otimes \boldsymbol{v}^2\right) = \operatorname{div} \mathbf{T} + \rho \boldsymbol{f}, \tag{1.11}$$

where $\mathbf{T}$ is defined as $\mathbf{T}^1 + \mathbf{T}^2$. Since $\rho^\alpha \boldsymbol{v}_d^\alpha$ are caused by diffusion, which is usually much slower then convection we neglect their contribution to the convective term and get

$$\begin{aligned}
&\rho^1 \boldsymbol{v}^1 \otimes \boldsymbol{v}^1 + \rho^2 \boldsymbol{v}^2 \otimes \boldsymbol{v}^2 = \\
&\qquad \left(\rho^1 \boldsymbol{v} - \rho^1 \boldsymbol{v}_d^1\right) \otimes \left(\boldsymbol{v} - \boldsymbol{v}_d^1\right) + \left(\rho^2 \boldsymbol{v} - \rho^2 \boldsymbol{v}_d^2\right) \otimes \left(\boldsymbol{v} - \boldsymbol{v}_d^2\right) \\
&\qquad\qquad \approx \rho^1 \boldsymbol{v} \otimes \boldsymbol{v} + \rho^2 \boldsymbol{v} \otimes \boldsymbol{v} = \rho \boldsymbol{v} \otimes \boldsymbol{v}.
\end{aligned}$$

Since $c = c^1 + c^2$ equation for concentrations are not independent, we take one of them (e. g. equation for c^2) and get

$$\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0, \\
\frac{\partial c^2}{\partial t} + \nabla c^2 \boldsymbol{v} &= \operatorname{div}(\mathbf{K} \nabla c^2), \\
\frac{\partial \rho \boldsymbol{v}}{\partial t} + \operatorname{div}\left(\rho \boldsymbol{v} \otimes \boldsymbol{v}\right) &= \operatorname{div} \mathbf{T} + \rho \boldsymbol{f}.
\end{aligned}$$

After relabeling c^2 as c, dividing balance of linear momentum by ρ and putting $\frac{1}{\rho}\mathbf{T} = -pI + \mathbf{S}$ we get the desired system of equations.

1.3 Existence result

In the article [1] existence of weak solution to the considered system is proven for broad class of stress tensors and diffusivity matrices in three spatial dimensions. Following assumptions on the constitutive relations are considered :

- $\mathbf{S}$ is continuous function of concentration and shear rate and $\boldsymbol{q}_c$ is continuous function of concentration, shear rate and gradient of concentration.
- For all $\mathbf{D}_1, \mathbf{D}_2 \in \mathbf{R}^{d \times d}_{sym}$ and all $c \in [0, 1]$

$$\left(\mathbf{S}\left(c, \mathbf{D}_1\right) - \mathbf{S}\left(c, \mathbf{D}_2\right)\right) \cdot \left(\mathbf{D}_1 - \mathbf{D}_2\right) > 0 \tag{1.12}$$

- There exist constants $C_1, C_2, C_3 \in (0, \infty)$ and function $\gamma_1 \in L^\infty([0,1])$, such that for all $c \in [0,1]$ and all $\mathbf{D} \in \mathbf{R}^{d\times d}_{sym}$

$$C_1 |\mathbf{D}|^r - C_3 \le \mathbf{S}(c, \mathbf{D}) \cdot \mathbf{D}, \qquad \mathbf{S}(c, \mathbf{D}) \le C_2 \gamma_1(c) |\mathbf{D}|^r + C_3. \tag{1.13}$$

- There exist positive constants C_4, C_5, such that for all $c \in [0,1]$ and all positive s

$$\begin{aligned} C_4 (1+s)^\beta |z|^2 &\le -q_c(c, z, s) \\ |q_c(c, z, s)| &\le C_5 \gamma_1(c) (1+s)^\beta |z| \end{aligned} \tag{1.14}$$

Before we define weak solution we have to introduce notation of subspaces of standard Sobolev space $W^{1,p}(\Omega)$ of vector valued functions,

$$\begin{aligned} W^{1,p}_n(\Omega) &= \{\boldsymbol{v} \in W^{1,p}(\Omega) \,|\, \mathrm{tr}(\boldsymbol{v}) \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}, \\ W^{1,p}_{\mathrm{div}}(\Omega) &= \{\boldsymbol{v} \in W^{1,p}(\Omega) \,|\, \mathrm{div}\, \boldsymbol{v} = 0 \text{ a. e. in } \Omega\}, \\ W^{1,p}_{n,\mathrm{div}}(\Omega) &= W^{1,p}_{\mathrm{div}}(\Omega) \cap W^{1,p}_n(\Omega). \end{aligned}$$

We also introduce notation for scalar Sobolev functions, which vanish on some part of the boundary

$$W^{1,p}_{\Gamma_D}(\Omega) = \{c \in W^{1,p}(\Omega) \,|\, \mathrm{tr}(c) = 0 \text{ on } \Gamma_D\}.$$

We will denote duals to the above spaces as follows :

$$\begin{aligned} W^{-1,p'}_n(\Omega) &= \left(W^{1,p}_n(\Omega)\right)^*, \\ W^{-1,p'}_{\mathrm{div}}(\Omega) &= \left(W^{1,p}_{\mathrm{div}}(\Omega)\right)^*, \\ W^{-1,p'}_{n,\mathrm{div}}(\Omega) &= \left(W^{1,p}_{n,\mathrm{div}}(\Omega)\right)^*. \end{aligned}$$

In the whole thesis we will denote the Bochner space of p-integrable functions on interval $[a,b]$ with values in some Banach space X, which is equipped with a norm $\|.\|_X$, in the following way :

$$L^p(a, b; X) = \left\{ g : [a,b] \to X \middle| g \text{ is measurable and } \int_a^b \|g\|^p_X < \infty \right\}.$$

For continuous functions on the interval $[a,b]$ with values in Banach space X with norm $\|.\|_X$ we will use the following notation :

$$\mathcal{C}(a, b; X) = \left\{ g : [a,b] \to X \middle| g \text{ is continuous on } [a,b] \right\}.$$

We will also need notation for duality between Sobolev spaces and their duals. For every functional $g \in W^{-1,p'}(\Omega)$ and function $f \in W^{1,p}(\Omega)$ will $\langle g; f \rangle_\Omega$ denote application of continuous functional g on function f.

Definition 1.1 *Let $\Omega \in \mathcal{C}^{1,1}$, $\alpha \in [0,\infty)$, $\boldsymbol{f} \in L^{r'}\left(0,T;W^{-1,r'}(\Omega)\right)$, let the function $\boldsymbol{v}_0$ belong to the space $L^2_{\boldsymbol{n},\mathrm{div}}(\Omega)$, the function c_0 fulfills inequality $0 \le c_0 \le 1$ and* (1.12) - (1.14) *are satisfied for $r > \frac{6}{5}$ and $-r \le 2\beta \le r$ and let numbers m, s, q be defined as*

$$m = \min\{r', \tfrac{5}{6}r\}, \quad q = \min\{2, \tfrac{2r}{r-2\beta}\}, \quad s = \min\{2, \tfrac{2r}{r+2\beta}\},$$

then we say that a triple of functions $\boldsymbol{v}$,c and p is a weak solution of the system (1.1) - (1.3) *with boundary conditions* (1.4) - (1.6) *and initial conditions $\boldsymbol{v}_0$, c_0, if*

$$\begin{aligned}
\boldsymbol{v} &\in L^r\left(0,T;W^{1,r}_{n,\mathrm{div}}(\Omega)\right) \cap L^{\frac{5}{3}r}\left(0,T;L^{\frac{5}{3}r}(\Omega)\right), \\
\boldsymbol{v} &\in \mathcal{C}_{weak}\left(0,T;L^2(\Omega)\right), \\
\mathrm{tr}(\boldsymbol{v}) &\in L^2\left(0,T;L^2(\partial\Omega)\right), \\
\boldsymbol{v}_{,t} &\in L^m\left(0,T;W^{-1,m}_n(\Omega)\right), \\
p &\in L^m\left(0,T;L^m(\Omega)\right), \\
c - c_b &\in L^q\left(0,T;W^{1,q}_{\Gamma_D}(\Omega)\right), \\
c_{,t} &\in L^{s'}\left(0,T;W^{-1,s'}_{\Gamma_D}(\Omega)\right), \\
\left(1+|\mathbf{D}(\boldsymbol{v})|^2\right)^{\frac{\beta}{2}} &\in L^2\left(0,T;L^2(\Omega)\right), \\
& \quad 0 \le c \le 1 \quad a.e.\ in\ Q,
\end{aligned}$$

and $\boldsymbol{v}$,c and p fulfills for all $\boldsymbol{\varphi} \in L^{m'}\left(0,T;W^{1,m'}_n(\Omega)\right)$ and $\psi \in L^s\left(0,T;W^{1,s}_n(\Omega)\right)$ following weak formulation

$$\begin{aligned}
\langle \boldsymbol{v}_{,t};\boldsymbol{\varphi}\rangle_\Omega + \int_\Omega \mathbf{S}\cdot\mathbf{D}(\boldsymbol{\varphi}) + \alpha\int_{\partial\Omega}\boldsymbol{v}\cdot\boldsymbol{\varphi} - \int_\Omega \boldsymbol{v}\otimes\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi} &= \int_\Omega p\,\mathrm{div}\,\boldsymbol{\varphi} + \langle \boldsymbol{f};\boldsymbol{\varphi}\rangle_\Omega, \\
\langle c_{,t};\psi\rangle_\Omega + \int_\Omega \boldsymbol{q}_c\cdot\nabla\psi - \int_\Omega c\boldsymbol{v}\cdot\nabla\psi &= 0,
\end{aligned}$$

and initial conditions are met in the following sense

$$\lim_{t\to 0+} \|c(t)-c_0\|_2 + \|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2 = 0.$$

Existence of such weak solution is proven in the article [1]. In this thesis we prove global regularity result for two dimensional system with simplified constitutive relations. We derive estimates on second spatial derivatives of velocity and concentration. With these estimates at our disposal we prove uniqueness of the regular weak solution in the class of weak solutions. These estimates can be also the first step in construction of the classical solution from weak solution.

Chapter 2

Regularity result

2.1 Studied system

In the main part of the work we will consider simplified system of equations. We will restrict ourselves to the case of two spatial dimensions and we will consider only simplified constitutive relations

$$\begin{aligned} \mathbf{S} &= \mathbf{S}(c, \mathbf{D}) = \nu(c)\mathbf{D}, \\ \boldsymbol{q}_c &= \boldsymbol{q}_c(\nabla c) = -k\nabla c, \end{aligned} \tag{2.1}$$

where k is positive real number, and viscosity $\nu : [0,1] \to \mathbf{R}$ is a positive Lipschitz function. We will restrict ourselves to Navier slip boundary condition for velocity, and zero concentration flux boundary condition for concentration, therefore

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{n} &= 0 \quad \text{on } \Gamma, \\ (\mathbf{S}\boldsymbol{n})_\tau &= 0 \quad \text{on } \Gamma, \\ \boldsymbol{q}_c \cdot \boldsymbol{n} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{2.2}$$

With these additional assumptions we get

$$\begin{aligned} \boldsymbol{v}_{,t} - \operatorname{div}(\nu(c)\mathbf{D}(\boldsymbol{v})) + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) &= -\nabla p + \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{v} &= 0, \\ c_{,t} - \operatorname{div}(k\nabla c) + \operatorname{div}(c\boldsymbol{v}) &= 0. \end{aligned} \tag{2.3}$$

We define the weak solution similarly as in 1.1, but since the constitutive relations were simplified and we now consider only two spatial dimensions, the definition has to be slightly modified.

Definition 2.1 *Let $\boldsymbol{f} \in L^2(0,T;W^{-1,2}(\Omega))$, $\boldsymbol{v}_0 \in L^2(\Omega)$ and c_0 fulfills $0 \leq c_0 \leq 1$, then triple of functions $\boldsymbol{v}, c$ a p is weak solution of the problem* (2.3) *with boundary conditions* (2.2) *and initial conditions $\boldsymbol{v}_0$ and c_0, if*

$$
\begin{aligned}
\boldsymbol{v} &\in L^2\left(0,T;W^{1,2}_{n,\mathrm{div}}\left(\Omega\right)\right)\cap\mathcal{C}\left(0,T;L^2\left(\Omega\right)\right)\\
\boldsymbol{v}_{,t} &\in L^2\left(0,T;W^{-1,2}_{n}\left(\Omega\right)\right)\\
c &\in L^2\left(0,T;W^{1,2}\left(\Omega\right)\right)\cap\mathcal{C}\left(0,T;L^2\left(\Omega\right)\right)\\
c_{,t} &\in L^2\left(0,T;W^{-1,2}\left(\Omega\right)\right)\\
p &\in L^2\left(0,T;L^2\left(\Omega\right)\right)
\end{aligned}
$$

and for all $\boldsymbol{\varphi}\in L^2\left(0,T;W^{1,2}_n\left(\Omega\right)\right)$ *and* $\psi\in L^2\left(0,T;W^{1,2}\left(\Omega\right)\right)$ *following equations are satisfied*

$$
\begin{aligned}
\langle\boldsymbol{v}_{,t};\boldsymbol{\varphi}\rangle_\Omega+\int_\Omega\nu\left(c\right)\mathbf{D}\left(\boldsymbol{v}\right)\cdot\mathbf{D}\left(\boldsymbol{\varphi}\right)-\int_\Omega\boldsymbol{v}\otimes\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi} &= \int_\Omega p\,\mathrm{div}\,\boldsymbol{\varphi}+\langle\boldsymbol{f};\boldsymbol{\varphi}\rangle_\Omega\\
\langle c_{,t};\psi\rangle_\Omega+k\int_\Omega\nabla c\cdot\nabla\psi-\int_\Omega c\boldsymbol{v}\cdot\nabla\psi &= 0
\end{aligned}
\tag{2.4}
$$

and initial conditions are attained in the following sense

$$
\lim_{t\to0+}\left\|c(t)-c_0\right\|_2+\left\|\boldsymbol{v}(t)-\boldsymbol{v}_0\right\|_2=0.
\tag{2.5}
$$

Adaptation of the existence result from [1] to two spatial dimensions and simplified boundary conditions and constitutive relations gives the existence of weak solution to the problem (2.1) - (2.3) in the sense of Definition 2.1.

Theorem 2.2 *Let* $\Omega\in\mathcal{C}^{1,1}$, $\boldsymbol{f}\in L^2\left(0,T;W^{-1,2}\left(\Omega\right)\right)$, $\boldsymbol{v}_0$ *belongs to space* $L^2\left(\Omega\right)$ *and* c_0 *satisfies* $0\le c_0\le 1$, *then there exists weak solution defined in 2.1 to the system* (2.3) *with boundary conditions* (2.2) *and initial conditions* $\boldsymbol{v}_0$ *and* c_0 *and there exists constant* C *dependent only on* c_0, $\boldsymbol{v}_0$, Ω, $\boldsymbol{f}$ *and* ν *such that*

$$
\begin{aligned}
&\left\|\boldsymbol{v}\right\|_{L^\infty(L^2)}+\left\|\boldsymbol{v}\right\|_{L^2(W^{1,2})}+\left\|c\right\|_{L^\infty(L^2)}+\left\|c\right\|_{L^2(W^{1,2})}+\left\|p\right\|_{L^2(L^2)}\\
&\qquad+\left\|\boldsymbol{v}_{,t}\right\|_{L^2(W^{-1,2})}+\left\|c_{,t}\right\|_{L^2(W^{-1,2})}\le C
\end{aligned}
\tag{2.6}
$$

The main part of the thesis is proof of following regularity theorem.

Theorem 2.3 *Let* $\boldsymbol{v}$, p *and* c *be weak solution of the problem* (2.1) - (2.3) *in the sense of definition 2.1 and let* $\nu:[0,1]\to\mathbf{R}$ *be Lipschitz continuous positive function if moreover initial condition* c_0 *is in space* $W^{1,2}\left(\Omega\right)$ *and* $\boldsymbol{v}_0$ *belongs to space* $W^{1,2}_n\left(\Omega\right)$, *and boundary of* Ω *is* $\mathcal{C}^4$, *then*

$$
c \in L^2\left(0,T;W^{2,2}\left(\Omega\right)\right)\cap L^\infty\left(0,T;W^{1,2}\left(\Omega\right)\right),
\tag{2.7}
$$

$$
\boldsymbol{v} \in L^2\left(0,T;W^{2,2}\left(\Omega\right)\right)\cap L^\infty\left(0,T;W^{1,2}\left(\Omega\right)\right).
\tag{2.8}
$$

Such estimate on second spatial derivatives of the velocity is well known for two dimensional Navier-Stokes equations. In case of Navier-Stokes flow in the plane the convective term has exactly the critical growth and can be estimated with interpolation inequality 2.9, which is not possible for three dimensional case where interpolation inequality 2.9 does not hold (only weaker estimate is possible) and regularity for three dimensional Navier-Stokes system is an open problem. Since we consider only two dimensional flow the convective term can be estimated with interpolation inequality, but further complication arises from dependence of viscosity on concentration.

With Theorem 2.3 at our disposal, uniqueness of weak solution can be obtained easily. In section 3.8 of the thesis we will establish the following result.

Theorem 2.4 *Let the assumptions of Theorem 2.3 be satisfied, then there exists a triple of functions* $\boldsymbol{v}$, c *and* p, *which is a weak solution of the problem* (2.1) - (2.3) *in the sense of definition 2.1 and this solution is an unique weak solution.*

2.2 Auxiliary lemmas

In the proof of the main theorem we will approximate partial derivatives with differences. We will need some basic properties of the differences listed below. For function g we define the difference $\mathrm{D}_s^h(g)$ in the direction $\boldsymbol{e}_s$ as

$$\mathrm{D}_s^h(g)(x) = \frac{1}{h}\left(g(x+h\boldsymbol{e}_s) - g(x)\right). \tag{2.9}$$

We will also use symbol $\nabla^h(g)$ defined as follows

$$\nabla^h(g) = \begin{pmatrix} \mathrm{D}_1^h(g) \\ \mathrm{D}_2^h(g) \end{pmatrix}.$$

We also define translation operator τ_h^s by the following relation,

$$\tau_h^s(g)(x) = g(x+h\boldsymbol{e}_s).$$

Lemma 2.5 *Let* A *be open subset of* $\mathbf{R}^n$ *and* $g : A \to \mathbf{R}$ *and let* $A' \subset A$ *be, such that* $\overline{A'} \subset A$ *and let* $p \in (1,\infty)$ *and there is some positive constant* C *such that*

$$\left\|\mathrm{D}_s^h(g)\right\|_{p,A'} \leq C \quad \textit{for all } h \leq \frac{1}{2}\mathrm{dist}\,(\partial A, A')$$

Then g *has weak partial derivative in the direction* e_s *and*

$$\left\|\frac{\partial g}{\partial x_s}\right\|_{p,A'} \leq C.$$

Lemma 2.6 *Let A be an open subset of $\mathbf{R}^n$ and $g : A \to \mathbf{R}$, if $\frac{\partial g}{\partial x_s} \in L^p(\Omega)$ for $p \in (1, \infty)$ then for all $h \leq \frac{1}{2}\mathrm{dist}(\partial A, A')$ function g satisfies the following estimate,*

$$\left\| \mathrm{D}_s^h(g) \right\|_{p,A'} \leq \left\| \frac{\partial g}{\partial x_s} \right\|_{p,A}.$$

Lemma 2.7 *Let A be open subset of $\mathbf{R}^n$ and $g : A \to \mathbf{R}$, if $\frac{\partial g}{\partial x_s} \in L^p(\Omega)$ for $p \in (1, \infty)$ then for every open $A' \subset A$, such that $\overline{A'} \subset A$*

$$\mathrm{D}_s^h(g) \to \frac{\partial g}{\partial x_s} \quad \textit{strongly in} \quad L^p(A')$$

Proofs of the above lemmas can be found in [5]. We will also need Korn inequality and interpolation inequality between spaces $L^2(\Omega)$ and $W^{1,2}(\Omega)$.

Lemma 2.8 (Korn's inequality) *Let $\boldsymbol{v} : \Omega \to \mathbf{R}^d$ be in the space $W^{1,2}(\Omega)$ then there exists constant C_K, which can depend on Ω, such that*

$$\|\boldsymbol{v}\|_{1,2} \leq C_K \left(\|\boldsymbol{v}\|_2 + \|\mathbf{D}(\boldsymbol{v})\|_2 \right).$$

Lemma 2.9 (Interpolation inequality) *Let $d = 2$ and $g : \Omega \to \mathbf{R}$ be in $L^2(\Omega) \cap W^{1,2}(\Omega)$ then there exists constant C such that*

$$\|g\|_4 \leq C \|g\|_{1,2}^{\frac{1}{2}} \|g\|_2^{\frac{1}{2}}.$$

Such interpolation inequality can be found in [8].

Lemma 2.10 *Let $[a, b] \subset \mathbf{R}$ be a bounded closed interval and let $g^n : [a, b] \to \mathbf{R}$ be a sequence of continuous nondecreasing functions, which pointwise converges to continuous nondecreasing function $g : [a, b] \to \mathbf{R}$, then g^n converges to g uniformly.*

Proof : Let $\epsilon > 0$ be arbitrary. Our aim is to find n_0, such that for all $n > n_0$

$$\sup_{x \in [a,b]} |g(x) - g^n(x)| \leq \epsilon$$

We first observe that we can find points $\{y_i \in [g(a), g(b)] | i \in \{0, 1, \ldots, l\}\}$ such that

$$\begin{aligned} y_0 &= g(a), \\ y_n &= g(b), \\ y_i &< y_{i+1}, \\ |y_i - y_{i+1}| &\leq \frac{\epsilon}{5}. \end{aligned}$$

Let x_i, $i = 0, \dots l$ be such that

$$\begin{aligned} g(x_i) &= y_i, \\ x_0 &= a, \\ x_l &= b. \end{aligned}$$

Since g^n converges pointwise we can find n_0, such that for all $n > n_0$ and all $i \in \{0, 1, \dots l\}$

$$|g^n(x_i) - g(x_i)| \le \frac{\epsilon}{5} \tag{2.10}$$

For arbitrary $x \in [a, b]$ we can find j such that $x \in [x_j, x_{j+1}]$ and estimate

$$|g(x) - g^n(x)| \le |g(x) - g(x_j)| + |g(x_j) - g^n(x_j)| + |g^n(x_j) - g^n(x)|$$

From definition of x_j and monotonicity of g follows that the first term can be estimated by $\frac{\epsilon}{5}$ the second term can be also estimated by $\frac{\epsilon}{5}$ from (2.10) and the last term can be estimated in the following way.

$$\begin{aligned} &|g^n(x_j) - g^n(x)| \le |g^n(x_j) - g^n(x_{j+1})| \\ &\qquad \le |g^n(x_j) - g(x_j)| + |g(x_j) - g(x_{j+1})| + |g(x_{j+1}) - g^n(x_{j+1})| \le \frac{3\epsilon}{5}, \end{aligned}$$

which finishes the proof.

□

Chapter 3

Proof of the main theorem

3.1 Formal estimates

Before we start with the proof of theorem (2.3) we will make formal estimates on derivatives of concentration and velocity. We will consider periodic boundary conditions and we will use $-\chi_{[0,t]}\,\Delta c$ as test function in the equation for concentration and $-\chi_{[0,t]}\,\Delta \boldsymbol{v}$ as test function in balance of linear momentum. Since $-\chi_{[0,t]}\,\Delta c$ and $-\chi_{[0,t]}\,\Delta \boldsymbol{v}$ are not allowed test functions the following estimate is not rigorous proof but serves only as a heuristic. We start with the estimate on the concentration.

$$-\int_0^t \langle c_{,t}; \Delta c\rangle_\Omega - k\int_0^t\int_\Omega \nabla c\cdot\nabla\,\Delta c = \int_0^t\int_\Omega \boldsymbol{v}\cdot\nabla c\,\Delta c$$

We use integration by parts on the left hand side and we estimate the convective term using Hölder's, interpolation and Young's inequality in the following way,

$$\begin{aligned}\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{dt}}\left\|c\right\|_2^2 + k\int_0^t \left\|\nabla^2 c\right\|_2^2 &\le \int_0^t \left\|\boldsymbol{v}\right\|_4 \left\|\nabla c\right\|_4 \left\|\nabla^2 c\right\|_2 \\ &\le \int_0^t \left\|\boldsymbol{v}\right\|_4 \left\|\nabla c\right\|_2^{\frac{1}{2}} \left\|\nabla c\right\|_{1,2}^{\frac{1}{2}} \left\|\nabla^2 c\right\|_2 \le C(\epsilon)\int_0^t (1+\left\|\boldsymbol{v}\right\|_4^4)\left\|\nabla c\right\|_2^2 + \epsilon\int_0^t \left\|\nabla^2 c\right\|_2^2.\end{aligned}$$

We can choose $\epsilon \le \frac{k}{2}$ and absorb the second gradient in the left side, use Gronwall's inequality, take supremum over t and get

$$\frac{1}{2}\left\|\nabla c\right\|_{L^\infty(L^2)}^2 + \frac{k}{2}\left\|\nabla^2 c\right\|_{L^2(L^2)}^2 \le C. \tag{3.1}$$

At this point we use $-\chi_{[0,t]}\,\Delta \boldsymbol{v}$ as the test function in the balance of linear momentum.

$$\begin{aligned}-\int_0^t \langle \boldsymbol{v}_{,t}; \Delta \boldsymbol{v}\rangle_\Omega - \int_0^t\int_\Omega \nu(c)\,\mathbf{D}(\boldsymbol{v})\cdot\mathbf{D}(\Delta\boldsymbol{v}) \\ = \int_0^t\int_\Omega [\nabla\boldsymbol{v}]\boldsymbol{v}\cdot\Delta\boldsymbol{v} - \int_0^t\int_\Omega p\,\mathrm{div}\,(\Delta\boldsymbol{v}) - \int_0^t\int_\Omega \boldsymbol{f}\cdot\Delta\boldsymbol{v}\end{aligned} \tag{3.2}$$

We estimate individual terms in (3.2) starting with the time derivative,

$$-\int_0^t \langle \boldsymbol{v}_{,t}; \Delta\, \boldsymbol{v}\rangle_\Omega = \frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{dt}} \|\nabla \boldsymbol{v}\|_2^2 = \frac{1}{2}\|\nabla \boldsymbol{v}(t)\|_2^2 - \frac{1}{2}\|\nabla \boldsymbol{v}(0)\|_2^2 .$$

We estimate elliptic term from bellow using Hölder's, interpolation, Korn's and Young's inequalities and get

$$\begin{aligned}
-\int_0^t \int_\Omega \nu(c)\,\mathbf{D}(\boldsymbol{v}) \cdot \mathbf{D}(\Delta\, c) &= \int_0^t \int_\Omega \nabla\left(\nu(c)\,\mathbf{D}(\boldsymbol{v})\right) \cdot \nabla \mathbf{D}(\boldsymbol{v}) \\
&\geq \kappa \int_0^t \int_\Omega |\mathbf{D}(\nabla \boldsymbol{v})|^2 - C\int_0^t |\nu'(c)|\,|\nabla c|\,|\mathbf{D}(\boldsymbol{v})|\,|\mathbf{D}(\nabla \boldsymbol{v})| \\
&\geq \kappa \int_0^t \|\mathbf{D}(\nabla \boldsymbol{v})\|_2^2 - C\int_0^t \|\nabla c\|_4 \|\nabla \boldsymbol{v}\|_4 \|\mathbf{D}(\nabla \boldsymbol{v})\|_2 \\
&\geq \kappa \int_0^t \|\mathbf{D}(\nabla \boldsymbol{v})\|_2^2 - C\int_0^t \|\nabla c\|_4 \|\nabla \boldsymbol{v}\|_2^{\frac{1}{2}} \|\nabla \boldsymbol{v}\|_{1,2}^{\frac{1}{2}} \|\mathbf{D}(\nabla \boldsymbol{v})\|_2 \\
&\geq \frac{\kappa}{C_k}\int_0^t \left\|\nabla^2 \boldsymbol{v}\right\|_2^2 - C - C(\epsilon)\int_0^t \left(1 + \|\nabla c\|_4^4\right) \|\nabla \boldsymbol{v}\|_2^2 - \epsilon \int_0^t \left\|\nabla^2 \boldsymbol{v}\right\|_2^2 .
\end{aligned}$$

We estimate the convective term as follows :

$$\begin{aligned}
\int_0^t \int_\Omega [\nabla \boldsymbol{v}]\boldsymbol{v} \cdot \Delta\, \boldsymbol{v} &\leq \int_0^t \|\nabla \boldsymbol{v}\|_4 \|\boldsymbol{v}\|_4 \left\|\nabla^2 \boldsymbol{v}\right\|_2 \\
&\leq \int_0^t \|\boldsymbol{v}\|_2^{\frac{1}{2}} \|\boldsymbol{v}\|_{1,2}^{\frac{1}{2}} \|\nabla \boldsymbol{v}\|_2^{\frac{1}{2}} \|\nabla \boldsymbol{v}\|_{1,2}^{\frac{1}{2}} \left\|\nabla^2 \boldsymbol{v}\right\|_2 \\
&\leq C(\epsilon)\int_0^t \|\nabla \boldsymbol{v}\|_2^2 \|\nabla \boldsymbol{v}\|_2^2 + \epsilon \int_0^t \left\|\nabla^2 \boldsymbol{v}\right\|_2^2 + C.
\end{aligned}$$

. Since $\operatorname{div} \boldsymbol{v} = 0$ the pressure term disappears and we estimate the term of volume force with Hölder's and Young's inequality in the following way :

$$\int_0^t \int_\Omega \boldsymbol{f} \cdot \Delta\, \boldsymbol{v} \leq \int_0^t \|\boldsymbol{f}\|_2 \left\|\nabla^2 \boldsymbol{v}\right\|_2 \leq C(\epsilon)\int_0^t \|\boldsymbol{f}\|_2^2 + \epsilon \int_0^t \left\|\nabla^2 \boldsymbol{v}\right\|_2^2 .$$

Now we put the individual terms together and get

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{v}(t)\|_2^2 + \frac{\kappa}{C_k}\int_0^t \left\|\nabla^2 \boldsymbol{v}\right\|_2^2 \\
&\qquad\qquad \leq C(\epsilon)\int_0^t (1 + \|\nabla c\|_4^4 + \|\nabla \boldsymbol{v}\|_2^2)\,\|\nabla \boldsymbol{v}\|_2^2 + 3\epsilon \int_0^t \left\|\nabla^2 \boldsymbol{v}\right\|_2^2 + C
\end{aligned}$$

We can choose ϵ small and absorb the second gradient from the left hand side in the right hand side, use Gronwall's inequality and take supremum over t to get

$$\frac{1}{2}\|\nabla \boldsymbol{v}\|_{L^\infty(L^2)}^2 + \frac{\kappa}{4C_k}\left\|\nabla^2 \boldsymbol{v}\right\|_{L^2(L^2)}^2 \leq C. \tag{3.3}$$

The estimates (3.1) and (3.3) are the results we wanted.

3.2 Outline of the proof

We will estimate the second derivatives by using second differences of c and $\boldsymbol{v}$ as test functions in equations for concentration and in balance of linear momentum. Since $\mathrm{D}_s^{-h}\left(\mathrm{D}_s^h(c)\right)$ and $\mathrm{D}_s^{-h}\left(\mathrm{D}_s^h(c)\right)$ are not well defined near the boundary of Ω we have to proceed more carefully. For every point $x_0 \in \overline{\Omega}$ we find some sufficiently small open neighborhood U_{x_0} of x_0 and prove that

$$\begin{aligned} c &\in L^2\left(0,T;W^{2,2}_{loc}\left(U_{x_0}\cap\Omega\right)\right)\cap L^\infty\left(0,T;W^{1,2}_{loc}\left(U_{x_0}\cap\Omega\right)\right),\\ \boldsymbol{v} &\in L^2\left(0,T;W^{2,2}_{loc}\left(U_{x_0}\cap\Omega\right)\right)\cap L^\infty\left(0,T;W^{1,2}_{loc}\left(U_{x_0}\cap\Omega\right)\right). \end{aligned} \tag{3.4}$$

We will prove (3.4) by flattening the boundary, and extending of the variables over the flat boundary. Since $\overline{\Omega}$ is compact and $\{U_{x_0}|x_0\in\Omega\}$ is its open cover, we can find finite open subcover $\mathcal{A}=\{U_x|x\in A\}$. Result (3.4) applied on sets from $\mathcal{A}$ and use of partition of unity gives the statement.

3.3 Flattening of the boundary

We assume, that the boundary of Ω can be in the neighborhood of $\boldsymbol{x}_0$ described as range of parametric curve $b:(-\gamma,\gamma)\to\mathbf{R}^2$, $t\in(-\gamma,\gamma)$,

$$b(t)=\begin{pmatrix} b^1(t)\\ b^2(t)\end{pmatrix}.$$

In the whole thesis we will assume that $b\in\mathcal{C}^4\left((-\gamma,\gamma)\right)$, $b(0)=\boldsymbol{x}_0$ and we will also assume that $\left|\dot{b}(t)\right|=1$, that is possible since we can reparametrize the curve b. We define mapping $T:(-\gamma,\gamma)\times(-\gamma,\gamma)\to\mathbf{R}^2$ with the following formula

$$T(x_1,x_2)=\begin{pmatrix} b^1(x_1)-\omega(x_1,x_2)\dot{b}^2(x_1)\\ b^2(x_1)+\omega(x_1,x_2)\dot{b}^1(x_1)\end{pmatrix}, \tag{3.5}$$

where ω will be chosen such that $\det(\nabla T)=1$. This choice of mapping T will allow us to extend the concentration and the velocity over the flattened boundary and handle the pressure term easily ($\det(\nabla T)=1$).

$$\nabla T(x_1,x_2)=\begin{pmatrix} \dot{b}^1(x_1)-\omega(x_1,x_2)\ddot{b}^2(x_1)-\frac{\partial\omega}{\partial x_1}(x_1,x_2)\dot{b}^2(x_1) & -\frac{\partial\omega}{\partial x_2}(x_1,x_2)\dot{b}^2(x_1)\\ \dot{b}^2(x_1)+\omega(x_1,x_2)\ddot{b}^1(x_1)+\frac{\partial\omega}{\partial x_1}(x_1,x_2)\dot{b}^1(x_1) & \frac{\partial\omega}{\partial x_2}(x_1,x_2)\dot{b}^1(x_1)\end{pmatrix}$$

The condition $\det(\nabla T)=1$ gives us the differential equation for ω in the following form :

$$\begin{aligned} \frac{\partial\omega}{\partial x_2}(x_1,x_2) &+ \frac{\partial\omega}{\partial x_2}(x_1,x_2)\omega(x_1,x_2)\left(\dot{b}^1(x_1)\ddot{b}^2(x_1)-\ddot{b}^1(x_1)\dot{b}^2(x_1)\right)=1\\ \frac{\partial\omega}{\partial x_2}(x_1,x_2) &= \frac{1}{1+\omega(x_1,x_2)\left(\dot{b}^1(x_1)\ddot{b}^2(x_1)-\ddot{b}^1(x_1)\dot{b}^2(x_1)\right)}. \end{aligned} \tag{3.6}$$

This equation is an ordinary differential equation with independent variable x_2 and parameter x_1. We equip the equation with initial condition $\omega(x_1, 0) = 0$ to ensure that the set $(-\gamma, \gamma) \times \{0\}$ is mapped to $\partial\Omega$. From theory for ordinary differential equations dependent on parameter (for example [6]) follows that there exists γ such that ω is defined on $(-\gamma, \gamma) \times (-\gamma, \gamma)$. Since $\left(\dot{b}^1(x_1)\ddot{b}^2(x_1) - \ddot{b}^1(x_1)\dot{b}^2(x_1)\right)$ is smooth of class $\mathcal{C}^2$, ω is also smooth of class $\mathcal{C}^2$.

Lemma 3.1 *Let mapping T be defined by formula (3.5), then following statements hold:*

1. *T is differentiable of class $\mathcal{C}^2$,*
2. $\det(\nabla T) = 1$,
3. *There exist $\epsilon_0 > 0$ such that for all positive $\epsilon < \epsilon_0$ T is $\mathcal{C}^2$ diffeomorphism between $(-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ and $T((-\epsilon, \epsilon) \times (-\epsilon, \epsilon))$,*
4. *$\nabla T(x)$ is orthogonal matrix for all $x \in (-\gamma, \gamma) \times \{0\}$,*
5. *For every $\delta > 0$ there exists $\epsilon_0 > 0$ such that for all positive $\epsilon < \epsilon_0$ there exist matrix functions $Q : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to \mathbf{R}^{2\times2}$ and $D : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to \mathbf{R}^{2\times2}$, such that $Q(x)$ is orthogonal for every $x \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ and*

$$\|D(x)\| \leq \delta \quad \text{for all} \quad x \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$$

   and

$$\nabla T(x) = Q(x) + D(x) \quad \text{for all} \quad x \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)\,. \tag{3.7}$$

Proof :

1. Follows from definition of ω.
2. Follows from definition of ω.
3. Follows immediately from part *1.* and *2.* and inverse function theorem.
4. Since ω has following properties

$$\begin{aligned} \omega(x_1, 0) &= 0 \quad \text{for all } x_1 \in (-\gamma, \gamma)\,, \\ \frac{\partial \omega}{\partial x_2}(x_1, 0) &= 1 \quad \text{for all } x_1 \in (-\gamma, \gamma)\,, \\ \frac{\partial \omega}{\partial x_1}(x_1, 0) &= 0 \quad \text{for all } x_1 \in (-\gamma, \gamma)\,, \end{aligned}$$

   for $x_2 = 0$ ∇T reduces to (3.8).

$$\nabla T(x_1, 0) = \begin{pmatrix} \dot{b}^1(x_1) & -\dot{b}^2(x_1) \\ \dot{b}^2(x_1) & \dot{b}^1(x_1) \end{pmatrix}. \tag{3.8}$$

   This matrix is the orthogonal matrix because $\left|\dot{b}(t)\right| = 1$.

5. We will find explicit formula for Q and D

$$Q(x_1,x_2) = \begin{pmatrix} \dot{b}^1(x_1) & -\dot{b}^2(x_1) \\ \dot{b}^2(x_1) & \dot{b}^1(x_1) \end{pmatrix}$$

$$D(x_1,x_2) = \begin{pmatrix} -\omega(x_1,x_2)\ddot{b}^2(x_1) - \frac{\partial \omega}{\partial x_1}(x_1,x_2)\dot{b}^2(x_1) & -(\frac{\partial \omega}{\partial x_2}(x_1,x_2) - 1)\dot{b}^2(x_1) \\ +\omega(x_1,x_2)\ddot{b}^1(x_1) + \frac{\partial \omega}{\partial x_1}(x_1,x_2)\dot{b}^1(x_1) & (\frac{\partial \omega}{\partial x_2}(x_1,x_2) - 1)\dot{b}^1(x_1) \end{pmatrix}$$

Statement immediately follows from smoothness of ω and properties of ω from the part 4.

$\square$

Lemma 3.2 *Let $\eta > 0$ be small enough to ensure injectivity of T on $(-\eta,\eta)\times(-\eta,\eta)$, then the following statements hold*

1. $T^{-1} : T\left((-\eta,\eta)\times(-\eta,\eta)\right) \to (-\eta,\eta)\times(-\eta,\eta)$ *is $\mathcal{C}^2$ diffeomorphism.*
2. $\det\left(\nabla T^{-1}(x)\right) = 1$ *for all $x \in T\left((-\eta,\eta)\times(-\eta,\eta)\right)$.*
3. $\nabla T^{-1}(x)$ *is orthogonal matrix for all $x \in \mathrm{Rng}\,(b) \cap T\left((-\eta,\eta)\times(-\eta,\eta)\right)$.*
4. *For every δ there exist ϵ_0 and matrix function $E : (-\eta,\eta)\times(-\eta,\eta) \to \mathbf{R}^{2\times 2}$ such that for all positive $\epsilon < \epsilon_0$*

$$\|E(x)\| \le \delta \qquad \textit{for all } x \in (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$$

and for Q from previous lemma holds

$$\nabla T^{-1}(x) = Q^T\left(T^{-1}(x)\right) + E\left(T^{-1}(x)\right) \qquad \textit{for all } x \in T\left((-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\right).$$

Proof :

1. An immediate consequence of Lemma 3.1 part *3.*
2. An immediate consequence of Lemma 3.1 part *2.*
3. An immediate consequence of Lemma 3.1 part *4.*
4. Follows from Lemma 3.1 part *5.* and continuity of T and matrix inversion.

$\square$

3.4 Transformation of the equations

For $\epsilon > 0$ we define G, G^+, G^-, G_0, U, U^+, U^-, U_0 as follows :

$$\begin{aligned} G &= (-\epsilon,\epsilon)\times(-\epsilon,\epsilon)\,, & G^+ &= (-\epsilon,\epsilon)\times(0,\epsilon)\,,\\ G^- &= (-\epsilon,\epsilon)\times(-\epsilon,0)\,, & G^0 &= (-\epsilon,\epsilon)\times\{0\},\\ U &= T(G), & U^+ &= T(G^+),\\ U^- &= T(G^-), & U^0 &= T(G^0). \end{aligned}$$

We can find ϵ_0 small enough that statements of Lemmas 3.1 and identities

$$\begin{aligned} U^+ &= U\cap\Omega,\\ U^- &= U\cap\overline{\Omega}^c,\\ U^0 &= \partial\Omega\cap U, \end{aligned} \tag{3.9}$$

hold for all $\epsilon < \epsilon_0$.

Since $\boldsymbol{v}$, c and p solves the equations on U^+ we can transform the equations to G with mapping T. We define functions

$$\begin{aligned} c'(y) &= c(T(y)) \quad \text{for all } y\in G^+,\\ p'(y) &= p(T(y)) \quad \text{for all } y\in G^+,\\ \boldsymbol{v}'(y) &= \nabla T^{-1}(T(y))\,\boldsymbol{v}(T(y)) \quad \text{for all } y\in G^+. \end{aligned} \tag{3.10}$$

We can compute inverse relations.

$$\begin{aligned} c(x) &= c'\left(T^{-1}(x)\right) \quad \text{for all } x\in U^+\\ p(x) &= p'\left(T^{-1}(x)\right) \quad \text{for all } x\in U^+\\ \boldsymbol{v}(x) &= \nabla T\left(T^{-1}(x)\right)\boldsymbol{v}'\left(T^{-1}(x)\right) \quad \text{for all } x\in U^+ \end{aligned}$$

In order to transform the equations we need to express partial derivatives of $\boldsymbol{v}$, c and p in terms of derivatives of $\boldsymbol{v}'$, c' and p' and T.

$$\frac{\partial c}{\partial x_i}(x) = \frac{\partial c'}{\partial y_k}\left(T^{-1}(x)\right)\frac{\partial T_k^{-1}}{\partial x_i}(x) \tag{3.11}$$

$$\frac{\partial p}{\partial x_i}(x) = \frac{\partial p'}{\partial y_k}\left(T^{-1}(x)\right)\frac{\partial T_k^{-1}}{\partial x_i}(x) \tag{3.12}$$

$$\frac{\partial \boldsymbol{v}_i}{\partial x_j}(x) = \frac{\partial T_i}{\partial y_k}\left(T^{-1}(x)\right)\frac{\partial \boldsymbol{v}'_k}{\partial y_l}\left(T^{-1}(x)\right)\frac{\partial T_l^{-1}}{\partial x_j}(x) \tag{3.13}$$

$$+\ \frac{\partial^2 T_i}{\partial y_l\partial y_k}\left(T^{-1}(x)\right)\frac{\partial T_k^{-1}}{\partial x_j}(x)\boldsymbol{v}'_l\left(T^{-1}(x)\right) \tag{3.14}$$

Now we transform individual terms of equation for concentration. We will assume in this section, that ψ has compact support in $U_0^+\times[0,T]$, $\psi\in L^2\left(0,T;W^{1,2}(\Omega)\right)$ and

$\psi' = \psi \circ T$. We will use integration by substitution with mapping T to pass from integrals over U to integrals over G. Since Jacobian of T is equal to one, integration by substitution reduces to changing variables.

Since T is time independent, it is simple to show that time derivative transforms as follows:

$$\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{dt}} c; \psi \right\rangle_\Omega = \int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{dt}} c'; \psi' \right\rangle_{G^+}. \tag{3.15}$$

Transformation in elliptic term is more difficult, because it contains derivatives with respect to x, we use (3.11) and get

$$\begin{aligned}
&\int_0^T \int_\Omega \nabla c(x) \cdot \nabla \psi(x) \mathrm{dx} \\
&\quad = \int_0^T \int_\Omega \frac{\partial c'}{\partial y_k} \left(T^{-1}(x)\right) \frac{\partial T_k^{-1}}{\partial x_i}(x) \frac{\partial \psi'}{\partial y_j} \left(T^{-1}(x)\right) \frac{\partial T_j^{-1}}{\partial x_i}(x) \,\mathrm{dx} \\
&\quad = \int_0^T \int_{G^+} \frac{\partial c'}{\partial y_k}(y) \frac{\partial T_k^{-1}}{\partial x_i}(T(y)) \frac{\partial \psi'}{\partial y_j}(y) \frac{\partial T_j^{-1}}{\partial x_i}(T(y)) \,\mathrm{dy} \\
&\quad = \int_0^T \int_{G^+} \alpha_{kj}(y) \frac{\partial c'}{\partial y_k}(y) \frac{\partial \psi'}{\partial y_j}(y) \,\mathrm{dy}.
\end{aligned} \tag{3.16}$$

In (3.16) α_{ij} is defined as

$$\alpha_{ij}(y) = \frac{\partial T_k^{-1}}{\partial x_i}(T(y)) \frac{\partial T_j^{-1}}{\partial x_i}(T(y)). \tag{3.17}$$

Since α_{ij} is product of matrix and its transposition, it is uniformly elliptic with respect to y. We transform the convective term as follows :

$$\begin{aligned}
&\int_0^T \int_\Omega \nabla c(x) \cdot \boldsymbol{v}(x) \psi(x) \mathrm{dx} \\
&\quad = \int_0^T \int_\Omega \frac{\partial c'}{\partial y_k} \left(T^{-1}(x)\right) \frac{\partial T_k^{-1}}{\partial x_i}(x) \frac{\partial T_i}{\partial y_l} \left(T^{-1}(x)\right) \boldsymbol{v}'_l \left(T^{-1}(x)\right) \psi' \left(T^{-1}(x)\right) \mathrm{dx} \\
&\quad = \int_0^T \int_\Omega \frac{\partial c'}{\partial y_k} \left(T^{-1}(x)\right) \delta_{kl} \boldsymbol{v}'_l \left(T^{-1}(x)\right) \psi' \left(T^{-1}(x)\right) \mathrm{dx} \\
&\quad = \int_0^T \int_\Omega \frac{\partial c'}{\partial y_k} \left(T^{-1}(x)\right) \boldsymbol{v}'_k \left(T^{-1}(x)\right) \psi' \left(T^{-1}(x)\right) \mathrm{dx} \\
&\quad = \int_0^T \int_{G^+} \frac{\partial c'}{\partial y_k}(y) \boldsymbol{v}'_k(y) \psi'(y) \,\mathrm{dy}.
\end{aligned} \tag{3.18}$$

To summarize, we obtained equation

$$\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{dt}} c'; \psi' \right\rangle_{G^+} + \int_0^T \int_{G^+} \alpha_{ij}(y) \frac{\partial c'}{\partial y_i}(y) \frac{\partial \psi'}{\partial y_j}(y)\, \mathrm{dy} \\ + \int_0^T \int_{G^+} \frac{\partial c'}{\partial y_k}(y)\, \boldsymbol{v}'_k(y)\, \psi'(y) \mathrm{dy} = 0, \quad (3.19)$$

which hold for every $\psi' \in L^2(0,T; W^{1,2}(G^+))$ with compact support in $G_0^+ \times [0,T]$.

In this section we transform individual terms in the balance of momentum. We will assume, that $\boldsymbol{\varphi}$ has compact support in $U_0^+ \times [0,T]$, $\boldsymbol{\varphi} \in L^2(0,T; W_n^{1,2}(\Omega))$, $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$ on the boundary of Ω and $\boldsymbol{\varphi}' = [\nabla T^{-1}] \circ T \psi \circ T$. It is important to note that, since $\boldsymbol{\varphi} \cdot \boldsymbol{n} = 0$ on the boundary of Ω we have $\boldsymbol{\varphi}'_2 = 0$ on G^0.

Time derivative transforms in the similar way as time derivative of the concentration :

$$\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{dt}} \boldsymbol{v}; \boldsymbol{\varphi} \right\rangle_{\Omega} = \int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{dt}} \boldsymbol{v}'; [\nabla T]^T \nabla T \boldsymbol{\varphi}' \right\rangle_{G^+}. \quad (3.20)$$

Symmetric gradient of velocity transforms as follows :

$$\begin{aligned} 2\left[\mathbf{D}(\boldsymbol{v})\right]_{ij}(T(y)) &= \frac{\partial \boldsymbol{v}_i}{\partial x_j}(T(y)) + \frac{\partial \boldsymbol{v}_i}{\partial x_j}(T(y)) \\ &= \frac{\partial T_i}{\partial y_k}(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y) \frac{\partial T_l^{-1}}{\partial x_j}(T(y)) + \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y) \frac{\partial T_k^{-1}}{\partial x_j}(T(y))\, \boldsymbol{v}'_l(y) \\ &\quad + \frac{\partial T_j}{\partial y_k}(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y) \frac{\partial T_l^{-1}}{\partial x_i}(T(y)) + \frac{\partial^2 T_j}{\partial y_l \partial y_k}(y) \frac{\partial T_k^{-1}}{\partial x_i}(T(y))\, \boldsymbol{v}'_l(y). \quad (3.21) \end{aligned}$$

We introduce the following notation :

$$\begin{aligned} \beta_{ijkl}(y) &= \frac{\partial T_i}{\partial y_k}(y) \frac{\partial T_l^{-1}}{\partial x_j}(T(y)) + \frac{\partial T_j}{\partial y_k}(y) \frac{\partial T_l^{-1}}{\partial x_i}(T(y)), \\ \gamma_{ijl}(y) &= \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y) \frac{\partial T_k^{-1}}{\partial x_j}(T(y)) + \frac{\partial^2 T_j}{\partial y_l \partial y_k}(y) \frac{\partial T_k^{-1}}{\partial x_i}(T(y)). \end{aligned}$$

Using notation above we express the elliptic term with derivatives of T and derivatives of

$\boldsymbol{v}$ with respect to y,

$$\begin{aligned}
\int_0^T \int_\Omega \nu\left(c(x)\right) \mathbf{D}\left(\boldsymbol{v}(x)\right) \cdot \mathbf{D}\left(\boldsymbol{\varphi}(x)\right) \mathrm{dx} \\
&= \frac{1}{4} \int_0^T \int_{G^+} \nu\left(c'(y)\right) \omega_{mnkl}(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y) \frac{\partial \boldsymbol{\varphi}'_m}{\partial y_n}(y) \, \mathrm{dy} \\
&+ \frac{1}{4} \int_0^T \int_{G^+} \nu\left(c'(y)\right) \sigma_{lmn}(y) \boldsymbol{v}'_l(y) \frac{\partial \boldsymbol{\varphi}'_m}{\partial y_n}(y) \, \mathrm{dy} \\
&+ \frac{1}{4} \int_0^T \int_{G^+} \nu\left(c'(y)\right) \sigma_{mkl}(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y) \, \boldsymbol{\varphi}'_m(y) \mathrm{dy} \\
&+ \frac{1}{4} \int_0^T \int_{G^+} \nu\left(c'(y)\right) \vartheta_{lm}(y) \boldsymbol{v}'_l(y) \boldsymbol{\varphi}'_m(y) \mathrm{dy},
\end{aligned}$$

where coefficients ω, σ and ϑ are defined as

$$\begin{aligned}
\omega_{mnkl}(y) &= \beta_{ijkl}(y)\beta_{ijmn}(y), \\
\sigma_{lmn}(y) &= \gamma_{ijl}(y)\beta_{ijmn}(y), \\
\vartheta_{lm}(y) &= \gamma_{ijl}(y)\gamma_{ijm}(y).
\end{aligned} \tag{3.22}$$

Now we transform the convective term, we use (3.13) and get

$$\begin{aligned}
\int_0^T \int_\Omega \nabla \boldsymbol{v}(x) \boldsymbol{v}(x) \cdot \boldsymbol{\varphi}(x) \mathrm{dx} \\
&= \int_0^T \int_{G^+} \frac{\partial T_i}{\partial y_k}(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y) \frac{\partial T_l^{-1}}{\partial x_j}\left(T(y)\right) \frac{\partial T_j}{\partial y_m}(y) \, \boldsymbol{v}'_m(y) \frac{\partial T_i}{\partial x_n}(y) \, \boldsymbol{\varphi}'_n(y) \, \mathrm{dy} \\
&+ \int_0^T \int_{G^+} \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y) \frac{\partial T_k^{-1}}{\partial x_j}\left(T(y)\right) \boldsymbol{v}'_l(y) \frac{\partial T_j}{\partial y_m}(y) \, \boldsymbol{v}'_m(y) \frac{\partial T_i}{\partial y_n}(y) \, \boldsymbol{\varphi}'_n(y) \, \mathrm{dy} \\
&= \int_0^T \int_{G^+} \frac{\partial T_i}{\partial y_k}(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y) \, \delta_{lm} \boldsymbol{v}'_m(y) \frac{\partial T_i}{\partial y_n}(y) \, \boldsymbol{\varphi}'_n(y) \, \mathrm{dy} \\
&+ \int_0^T \int_{G^+} \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y) \, \boldsymbol{v}'_l(y) \, \delta_{km} \boldsymbol{v}'_m(y) \frac{\partial T_i}{\partial y_n}(y) \, \boldsymbol{\varphi}'_n(y) \, \mathrm{dy} \\
&= \int_0^T \int_{G^+} \frac{\partial T_i}{\partial y_k}(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y) \, \boldsymbol{v}'_l(y) \frac{\partial T_i}{\partial y_n}(y) \, \boldsymbol{\varphi}'_n(y) \, \mathrm{dy} \\
&+ \int_0^T \int_{G^+} \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y) \, \boldsymbol{v}'_l(y) \, \boldsymbol{v}'_k(y) \frac{\partial T_i}{\partial y_n}(y) \, \boldsymbol{\varphi}'_n(y) \, \mathrm{dy} \\
&= \int_0^T \int_{G^+} \pi_{kn}(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y) \, \boldsymbol{v}'_l(y) \, \boldsymbol{\varphi}'_n(y) \, \mathrm{dy} \\
&+ \int_0^T \int_{G^+} \lambda_{lkn}(y) \boldsymbol{v}'_l(y) \, \boldsymbol{v}'_k(y) \, \boldsymbol{\varphi}'_n(y) \, \mathrm{dy},
\end{aligned}$$

where λ and π are defined by (3.23).

$$\pi_{kn}(y) = \frac{\partial T_i}{\partial y_k}(y)\frac{\partial T_i}{\partial y_n}(y)$$
$$\lambda_{lkn}(y) = \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\frac{\partial T_i}{\partial y_n}(y) \tag{3.23}$$

In the transformation of pressure term we use the fact that $\det(\nabla T) = 1$.

$$\begin{aligned}\int_0^T \int_\Omega p(x)\operatorname{div}\boldsymbol{\varphi}(x)\mathrm{dx} &= \int_0^T \int_{G^+} p'(y)\frac{\partial T_i}{\partial y_k}(y)\frac{\partial \boldsymbol{\varphi}'_k}{\partial y_l}(y)\frac{\partial T_l^{-1}}{\partial x_i}(T(y))\,\mathrm{dy} \\ &+ \int_0^T \int_{G^+} p'(y)\frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\frac{\partial T_k^{-1}}{\partial x_i}(T(y))\,\boldsymbol{\varphi}'_l(y)\,\mathrm{dy}\end{aligned}$$

By direct calculation we show that the second term is equal to zero :

$$\begin{aligned}&\frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\frac{\partial T_k^{-1}}{\partial x_i}(T(y)) = \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\left[\nabla T^{-1}(T(y))\right]_{li} = \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\left[\nabla T(y)\right]_{li}^{-1} \\ &= \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\begin{pmatrix}\frac{\partial T_2}{\partial x_2}(y) & -\frac{\partial T_1}{\partial x_2}(y) \\ \frac{\partial T_2}{\partial x_1}(y) & \frac{\partial T_1}{\partial x_1}(y)\end{pmatrix}_{li} = \frac{\partial^2 T_1}{\partial y_1 \partial y_k}(y)\frac{\partial T_2}{\partial x_2}(y) + \frac{\partial^2 T_2}{\partial y_2 \partial y_k}(y)\frac{\partial T_1}{\partial x_1}(y) \\ &- \frac{\partial^2 T_2}{\partial y_1 \partial y_k}(y)\frac{\partial T_1}{\partial x_2}(y) - \frac{\partial^2 T_1}{\partial y_2 \partial y_k}(y)\frac{\partial T_2}{\partial x_1}(y) = \frac{\partial}{\partial y_k}\left(\frac{\partial T_1}{\partial y_1}(y)\frac{\partial T_2}{\partial x_2}(y) - \frac{\partial T_2}{\partial y_1}(y)\frac{\partial T_1}{\partial x_2}(y)\right) \\ &= \frac{\partial}{\partial y_k}\left(\det\left(\nabla T(y)\right)\right) = 0.\end{aligned}$$

Therefore the the pressure term does not change its form and we get

$$\int_0^T \int_\Omega p(x)\operatorname{div}\boldsymbol{\varphi}(x)\mathrm{dx} = \int_0^T \int_{G^+} p'(y)\frac{\partial \boldsymbol{\varphi}'_i}{\partial y_i}(y)\,\mathrm{dy}.$$

Transformation in the force term is simple :

$$\int_0^T \int_\Omega \boldsymbol{f}(x)\cdot\boldsymbol{\varphi}(x)\mathrm{dx} = \int_0^T \int_{G^+} \frac{\partial T_i}{\partial x_k}(y)\,\boldsymbol{f}'_k(y)\frac{\partial T_i}{\partial x_l}(y)\,\boldsymbol{\varphi}'_l(y)\mathrm{dy}.$$

We put all terms together and we obtain the following equation :

$$
\begin{aligned}
&\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}'; [\nabla T]^T \nabla T \boldsymbol{\varphi}' \right\rangle_{G^+} \\
&\qquad + \frac{1}{4}\int_0^T \int_{G^+} \nu\left(c'(y)\right)\omega_{mnkl}(y)\frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y)\,\frac{\partial \boldsymbol{\varphi}'_m}{\partial y_n}(y)\,\mathrm{dy} \\
&\qquad + \frac{1}{4}\int_0^T \int_{G^+} \nu\left(c'(y)\right)\sigma_{lmn}(y)\boldsymbol{v}'_l(y)\frac{\partial \boldsymbol{\varphi}'_m}{\partial y_n}(y)\,\mathrm{dy} \\
&\qquad + \frac{1}{4}\int_0^T \int_{G^+} \nu\left(c'(y)\right)\sigma_{mkl}(y)\frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y)\,\boldsymbol{\varphi}'_m(y)\mathrm{dy} \\
&\qquad + \frac{1}{4}\int_0^T \int_{G^+} \nu\left(c'(y)\right)\vartheta_{lm}(y)\boldsymbol{v}'_l(y)\boldsymbol{\varphi}'_m(y)\mathrm{dy} \\
&\qquad + \int_0^T \int_{G^+} \pi_{kn}(y)\frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y)\,\boldsymbol{v}'_l(y)\,\boldsymbol{\varphi}'_n(y)\,\mathrm{dy} \\
&\qquad + \int_0^T \int_{G^+} \lambda_{lkn}(y)\boldsymbol{v}'_l(y)\,\boldsymbol{v}'_k(y)\,\boldsymbol{\varphi}'_n(y)\,\mathrm{dy} \\
&= \int_0^T \int_{G^+} p'(y)\frac{\partial \boldsymbol{\varphi}'_i}{\partial y_i}(y)\,\mathrm{dy} + \int_0^T \int_{G^+} \pi_{kl}(y)\,(y)\,\boldsymbol{f}'_k(y)\,(y)\,\boldsymbol{\varphi}'_l(y)\mathrm{dy}. \quad (3.24)
\end{aligned}
$$

This equation holds for all $\boldsymbol{\varphi}' \in L^2\left(0,T; W_n^{1,2}\left(G^+\right)\right)$ with compact support in $U_0^+ \times [0,T]$.

3.5 Extension of variables

In this section we extend the variables and coefficients from equation (3.24) and (3.19) from G^+ to the whole set G. We extent the concentration and second component of the velocity to be odd in y_2 variable and first component of the velocity to be even in y_2 variable, therefore

$$
\begin{aligned}
c'(y) &= c'(Sy) \quad \text{for all } y \in G^-, \\
p'(y) &= p'(Sy) \quad \text{for all } y \in G^-, \\
\boldsymbol{v}'(y) &= S\boldsymbol{v}'(Sy) \quad \text{for all } y \in G^-.
\end{aligned}
$$

Matrix S is defined as follows :

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.25)$$

This extension has the property that for c' and $\boldsymbol{v}'$ smooth and satisfying the boundary conditions, c' and $\boldsymbol{v}'$ have continuous first derivatives on the set G. It is obvious that without this property we would have no chance to prove the result. It is simple to show that $\boldsymbol{v}'$, p' and c' can be defined on G^- by using $R = T \circ S$ instead of T in (3.10), therefore

$$\begin{aligned} c'(y) &= c(R(y)) \quad \text{for all } y \in G^+, \\ p'(y) &= p(R(y)) \quad \text{for all } y \in G^+, \\ \boldsymbol{v}'(y) &= \nabla R^{-1}(R(y))\, \boldsymbol{v}(R(y)) \quad \text{for all } y \in G^+. \end{aligned} \tag{3.26}$$

We can transform (2.3) with diffeomorphism R to be system of equations on G^- in the same way as we did with mapping T and obtain almost the same system of equations, the only difference being usage of R instead of T in the definition of coefficients. We sum transformed weak formulations of the equations on G^+ and G^-, and we will handle that sum as if it was single equation on whole set G with coefficients defined as follows :

$$\begin{aligned}
\alpha_{ij}(y) &= \begin{cases} \frac{\partial T_k^{-1}}{\partial x_i}(T(y))\, \frac{\partial T_j^{-1}}{\partial x_i}(T(y)) & \text{for all } y \in G^+ \\ \frac{\partial R_k^{-1}}{\partial x_i}(R(y))\, \frac{\partial R_j^{-1}}{\partial x_i}(R(y)) & \text{for all } y \in G^- \end{cases} \\
\beta_{ijkl}(y) &= \begin{cases} \frac{\partial T_i}{\partial y_k}(y)\, \frac{\partial T_l^{-1}}{\partial x_j}(T(y)) + \frac{\partial T_j}{\partial y_k}(y)\, \frac{\partial T_l^{-1}}{\partial x_i}(T(y)) & \text{for all } y \in G^+ \\ \frac{\partial R_i}{\partial y_k}(y)\, \frac{\partial R_l^{-1}}{\partial x_j}(R(y)) + \frac{\partial R_j}{\partial y_k}(y)\, \frac{\partial R_l^{-1}}{\partial x_i}(R(y)) & \text{for all } y \in G^- \end{cases} \\
\gamma_{ijl}(y) &= \begin{cases} \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\, \frac{\partial T_k^{-1}}{\partial x_j}(T(y)) + \frac{\partial^2 T_j}{\partial y_l \partial y_k}(y)\, \frac{\partial T_k^{-1}}{\partial x_i}(T(y)) & \text{for all } y \in G^+ \\ \frac{\partial^2 R_i}{\partial y_l \partial y_k}(y)\, \frac{\partial R_k^{-1}}{\partial x_j}(R(y)) + \frac{\partial^2 R_j}{\partial y_l \partial y_k}(y)\, \frac{\partial R_k^{-1}}{\partial x_i}(R(y)) & \text{for all } y \in G^- \end{cases} \\
\pi_{kn}(y) &= \begin{cases} \frac{\partial T_i}{\partial y_k}(y)\, \frac{\partial T_i}{\partial y_n}(y) & \text{for all } y \in G^+ \\ \frac{\partial R_i}{\partial y_k}(y)\, \frac{\partial R_i}{\partial y_n}(y) & \text{for all } y \in G^- \end{cases} \\
\lambda_{mnkl}(y) &= \begin{cases} \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\, \frac{\partial T_i}{\partial y_n}(y) & \text{for all } y \in G^+ \\ \frac{\partial^2 R_i}{\partial y_l \partial y_k}(y)\, \frac{\partial R_i}{\partial y_n}(y) & \text{for all } y \in G^- \end{cases} \\
\omega_{mnkl}(y) &= \beta_{ijkl}(y)\beta_{ijmn}(y) \quad \text{for all } y \in G \\
\sigma_{lmn}(y) &= \gamma_{ijl}(y)\beta_{ijmn}(y) \quad \text{for all } y \in G \\
\vartheta_{lm}(y) &= \gamma_{ijl}(y)\gamma_{ijm}(y) \quad \text{for all } y \in G
\end{aligned}$$

We obtained that $\boldsymbol{v}'$, c' and p' satisfy equations (3.27) and (3.28) for every $\boldsymbol{\varphi}' \in L^2\left(0,T; W_n^{1,2}(G)\right)$ with compact support in $G \times [0,T]$ which have zero trace of its second component on $G^0 \times [0,T]$ and for every $\psi' \in L^2\left(0,T; W^{1,2}(G)\right)$ with compact support in $G \times [0,T]$.

$$\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{dt}} c'; \psi' \right\rangle_G + \int_0^T \int_G \alpha_{ij}(y) \frac{\partial c'}{\partial y_i}(y)\, \frac{\partial \psi'}{\partial y_j}(y)\, \mathrm{dy} + \int_0^T \int_G \frac{\partial c'}{\partial y_k}(y)\, \boldsymbol{v}_k'(y)\, \psi'(y) \mathrm{dy} = 0 \tag{3.27}$$

$$\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}'; [\nabla T]^T \nabla T \boldsymbol{\varphi}' \right\rangle_G + \frac{1}{4}\int_0^T \int_G \nu\left(c'(y)\right)\omega_{mnkl}(y)\frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y)\,\frac{\partial \boldsymbol{\varphi}'_m}{\partial y_n}(y)\,\mathrm{dy}$$
$$+\frac{1}{4}\int_0^T \int_G \nu\left(c'(y)\right)\sigma_{lmn}(y)\boldsymbol{v}'_l(y)\frac{\partial \boldsymbol{\varphi}'_m}{\partial y_n}(y)\,\mathrm{dy}$$
$$+\frac{1}{4}\int_0^T \int_G \nu\left(c'(y)\right)\sigma_{mkl}(y)\frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y)\,\boldsymbol{\varphi}'_m(y)\mathrm{dy}$$
$$+\frac{1}{4}\int_0^T \int_G \nu\left(c'(y)\right)\vartheta_{lm}(y)\boldsymbol{v}'_l(y)\boldsymbol{\varphi}'_m(y)\mathrm{dy}$$
$$+\int_0^T \int_G \pi_{kn}(y)\frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y)\,\boldsymbol{v}'_l(y)\,\boldsymbol{\varphi}'_n(y)\,\mathrm{dy}$$
$$+\int_0^T \int_G \lambda_{lkn}(y)\boldsymbol{v}'_l(y)\,\boldsymbol{v}'_k(y)\,\boldsymbol{\varphi}'_n(y)\,\mathrm{dy}$$
$$=\int_0^T \int_G p'(y)\frac{\partial \boldsymbol{\varphi}'_i}{\partial y_i}(y)\,\mathrm{dy} + \int_0^T \int_G \pi_{kl}(y)\,(y)\,\boldsymbol{f}'_k(y)\,(y)\,\boldsymbol{\varphi}'_l(y)\mathrm{dy} \quad (3.28)$$

In order to estimate second derivatives of $\boldsymbol{v}'$, c' we will need Lipschitz continuity of some of the coefficients. And some form of ellipticity in the highest order term.

Lemma 3.3 *Coefficients α, ω, π are Lipschitz continuous on G*

Proof : Since all of the coefficients are Lipschitz continuous on G^+ and on G^-, in order to obtain Lipschitz continuity on G it is enough to show, that the coefficients are continuous on G^0. For continuity on G^0 is enough to show that definitions of coefficients in G^+ and in G^- coincide on G^0.

α **:** Coefficient α_{ij} can be interpreted as the element in i-th row and j-th column of the matrix $\nabla T^{-1}(y)\left[\nabla T^{-1}(y)\right]^T$ in G^+ and $\nabla R^{-1}(y)\left[\nabla R^{-1}(y)\right]^T$ in G^-. For $y \in G^0$ coefficient $\alpha_{ij} = \delta_{ij}$ in both definitions, because matrices $\nabla T(y)$ and $\nabla R(y)$ are orthogonal for all $y \in G^0$ by lemma 3.1.

ω **:** For $y \in G^0$

$$4\omega_{klmn} = \left(\frac{\partial T_i}{\partial y_k}(y)\,\frac{\partial T_l^{-1}}{\partial x_j}(T(y)) + \frac{\partial T_j}{\partial y_k}(y)\,\frac{\partial T_l^{-1}}{\partial x_i}(T(y))\right)$$
$$\left(\frac{\partial T_i}{\partial y_m}(y)\,\frac{\partial T_n^{-1}}{\partial x_j}(T(y)) + \frac{\partial T_j}{\partial y_m}(y)\,\frac{\partial T_n^{-1}}{\partial x_i}(T(y))\right)$$
$$= \delta_{km}\delta_{ln} + \delta_{lm}\delta_{kn} + \delta_{kn}\delta_{lm} + \delta_{km}\delta_{ln} \quad (3.29)$$

Only fact we used was that $\nabla T(y)$ is orthogonal, since $\nabla R(y)$ is also orthogonal we get the same result for the definition of ω on G^-.

π **:** Similar proof as for α.

$\square$

Lemma 3.4 *Function $\sigma_{lmn}\boldsymbol{v}'_l$ is in space $L^2\left(0,T;W^{1,2}\left(G\right)\right)$*

Proof : Since σ_{lmn} is Lipschitz continuous on both G^+ and G^- and $\boldsymbol{v}'_l$ is in spaces $L^2\left(0,T;W^{1,2}\left(G^-\right)\right)$ and $L^2\left(0,T;W^{1,2}\left(G^+\right)\right)$ it is sufficient to show that traces of $\sigma_{lmn}\boldsymbol{v}'_l$ as function from spaces $L^2\left(0,T;W^{1,2}\left(G^-\right)\right)$ and $L^2\left(0,T;W^{1,2}\left(G^+\right)\right)$ are equal on $G^0\times[0,T]$.

$$\begin{aligned}\sigma_{lmn}(y) = \gamma_{ijl}(y)\beta_{ijmn}(y)&\\ = \left(\frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\frac{\partial T_k^{-1}}{\partial x_j}(T(y)) + \frac{\partial^2 T_j}{\partial y_l \partial y_k}(y)\frac{\partial T_k^{-1}}{\partial x_i}(T(y))\right)&\\ \left(\frac{\partial T_i}{\partial y_m}(y)\frac{\partial T_n^{-1}}{\partial x_j}(T(y)) + \frac{\partial T_j}{\partial y_m}(y)\frac{\partial T_n^{-1}}{\partial x_i}(T(y))\right)&\end{aligned} \tag{3.30}$$

We assume $y\in G^0$, therefore ∇T and ∇T^{-1} are orthogonal matrices. We can use this to simplify (3.30) :

$$\begin{aligned}&\left(\frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\frac{\partial T_k^{-1}}{\partial x_j}(T(y)) + \frac{\partial^2 T_j}{\partial y_l \partial y_k}(y)\frac{\partial T_k^{-1}}{\partial x_i}(T(y))\right)\\ &\qquad\left(\frac{\partial T_i}{\partial y_m}(y)\frac{\partial T_n^{-1}}{\partial x_j}(T(y)) + \frac{\partial T_j}{\partial y_m}(y)\frac{\partial T_n^{-1}}{\partial x_i}(T(y))\right)\\ &= \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\frac{\partial T_i}{\partial y_m}(y)\,\delta_{kn} + \frac{\partial^2 T_i}{\partial y_l \partial y_k}(y)\frac{\partial T_n^{-1}}{\partial x_i}(T(y))\,\delta_{km}\\ &+ \frac{\partial^2 T_j}{\partial y_l \partial y_k}(y)\frac{\partial T_n^{-1}}{\partial x_j}(T(y))\,\delta_{km} + \frac{\partial^2 T_j}{\partial y_l \partial y_k}(y)\frac{\partial T_j}{\partial y_m}(y)\,\delta_{kn}\\ &= \frac{\partial^2 T_i}{\partial y_l \partial y_n}(y)\frac{\partial T_i}{\partial y_m}(y) + \frac{\partial^2 T_i}{\partial y_l \partial y_m}(y)\frac{\partial T_i}{\partial y_n}(y)\\ &+ \frac{\partial^2 T_j}{\partial y_l \partial y_m}(y)\frac{\partial T_j}{\partial y_n}(y) + \frac{\partial^2 T_j}{\partial y_l \partial y_n}(y)\frac{\partial T_j}{\partial y_m}(y)\\ &= 2\left(\frac{\partial^2 T_i}{\partial y_l \partial y_n}(y)\frac{\partial T_i}{\partial y_m}(y) + \frac{\partial^2 T_j}{\partial y_l \partial y_m}(y)\frac{\partial T_j}{\partial y_n}(y)\right).\end{aligned}$$

Now we use the fact that $\boldsymbol{v}'_2$ has zero trace on $G^0\times[0,T]$ and fact that partial derivatives with respect to y_1 of ∇T depend only on values of ∇T on $G^0\times[0,T]$ and get

$$\begin{aligned}\sigma_{lmn}(y)\boldsymbol{v}'_l &= 2\boldsymbol{v}'_1\left(\frac{\partial^2 T_i}{\partial y_1 \partial y_n}(y)\frac{\partial T_i}{\partial y_m}(y) + \frac{\partial^2 T_i}{\partial y_1 \partial y_m}(y)\frac{\partial T_i}{\partial y_n}(y)\right)\\ &= 2\boldsymbol{v}'_1\frac{\partial}{\partial y_1}\left(\frac{\partial T_i}{\partial y_m}(y)\frac{\partial T_i}{\partial y_n}(y)\right) = 2\boldsymbol{v}'_1\frac{\partial}{\partial y_1}\left(\delta_{mn}+\delta_{mn}\right) = 0.\end{aligned}$$

Since exactly the same calculation can be made with the mapping R instead of T, the statement of the lemma holds.

$\square$

Lemma 3.5 *There exist constant μ, such that for every $\boldsymbol{\xi} \in \mathbf{R}^d$ and every $y \in G$ following inequality is satisfied*

$$\alpha_{ij}(y)\boldsymbol{\xi}_i\boldsymbol{\xi}_j \geq \mu \left|\boldsymbol{\xi}\right|^2 .$$

Proof : Estimate is direct consequence of α being product of matrix ∇T and its transpose and lemma 3.1.

$\square$

Lemma 3.6 *For ω defined above there exist $\epsilon > 0$ such that for all $\boldsymbol{u} \in W^{1,2}(G)$ with compact support in G inequality (3.31) holds.*

$$\int_G \nu\left(c'(y)\right)\omega_{mnkl}(y)\frac{\partial \boldsymbol{u}_k}{\partial y_l}(y)\,\frac{\partial \boldsymbol{u}_m}{\partial y_n}(y)\,\mathrm{dy} \geq \frac{\kappa}{2C_K}\int_G \left|\nabla \boldsymbol{u}\right|^2 - C\left\|\boldsymbol{u}\right\|_2^2, \tag{3.31}$$

where κ is positive number, such that

$$\kappa \leq \nu\left(c\right) \qquad \text{for all } c \in [0,1]. \tag{3.32}$$

Proof : Prior to the prove we note, that there exist κ, which satisfies 3.32, since ν is assumed continuous and interval [0,1] is compact set.

$$\begin{aligned}
4\int_G \nu\left(c'(y)\right)\omega_{mnkl}(y)\frac{\partial \boldsymbol{u}_k}{\partial y_l}(y)\,\frac{\partial \boldsymbol{u}_m}{\partial y_n}(y)\,\mathrm{dy} = \\
\int_G \nu\left(c'(y)\right)\left(\frac{\partial T_i}{\partial y_k}(y)\frac{\partial \boldsymbol{u}_k}{\partial y_l}(y)\frac{\partial T_l^{-1}}{\partial x_j}\left(T(y)\right) + \frac{\partial T_j}{\partial y_k}(y)\frac{\partial \boldsymbol{u}_k}{\partial y_l}(y)\frac{\partial T_l^{-1}}{\partial x_i}\left(T(y)\right)\right) \\
\left(\frac{\partial T_i}{\partial y_k}(y)\frac{\partial \boldsymbol{u}_k}{\partial y_l}(y)\frac{\partial T_l^{-1}}{\partial x_j}\left(T(y)\right) + \frac{\partial T_j}{\partial y_k}(y)\frac{\partial \boldsymbol{u}_k}{\partial y_l}(y)\frac{\partial T_l^{-1}}{\partial x_i}\left(T(y)\right)\right)\mathrm{dy}
\end{aligned} \tag{3.33}$$

We use decomposition of ∇T and ∇T^{-1} introduced in lemmas 3.1 and 3.2.

$$\begin{aligned}
&\int_G \nu\left(c'(y)\right)\left|(Q+D)(y)\nabla\boldsymbol{u}(y)(Q^T+E)(y) + \left((Q+D)(y)\nabla\boldsymbol{u}(y)(Q^T+E)(y)\right)^T\right|^2 \\
&= \int_G \nu\left(c'(y)\right)\left|(Q+D)(y)\nabla\boldsymbol{u}(y)(Q^T+E)(y) + (Q+E^T)(y)\left(\nabla\boldsymbol{u}(y)\right)^T(Q^T+D^T)(y)\right|^2 \\
&\qquad\qquad \geq \int_G \nu\left(c'(y)\right)\left|Q(y)\left(\nabla\boldsymbol{u}(y) + \left(\nabla\boldsymbol{u}(y)\right)^T\right)Q^T\right|^2 - \delta\left\|\nabla\boldsymbol{u}\right\|_2
\end{aligned}$$

We used lemmas 3.1 and 3.2 to estimate E and D. Since Q is orthogonal matrix, absolute value in the first term remains the same, if we remove Q and Q^T.

$$\int_G \nu\left(c'(y)\right)\left|\nabla\boldsymbol{u}(y) + \left(\nabla\boldsymbol{u}(y)\right)^T\right|^2 - \delta\left\|\nabla\boldsymbol{u}\right\|_2 \geq 4\kappa\int_G \left|\mathbf{D}\left(\boldsymbol{u}\right)\right|^2 - \delta\left\|\nabla\boldsymbol{u}\right\|_2$$

In this point we use Korn's inequality on the square G, it is important that constant C_K from Korn's inequality is independent of size of G (independent of ϵ).

$$4\kappa \int_G |\mathbf{D}(\boldsymbol{u})|^2 - \delta \|\nabla \boldsymbol{u}\|_2 \geq \left(\frac{4\kappa}{C_K} - \delta\right) \int_G |\nabla \boldsymbol{u}|^2 - C \|\boldsymbol{u}\|_2^2$$

We choose $\delta = \frac{2\kappa}{C_K}$, and find ϵ from lemmas 3.1 and 3.2 we get the the desired result.

□

3.6 Estimates on the second derivatives of concentration

Lemma 3.7 *Let $\boldsymbol{v}'$, c', p' satisfies (3.27) and (3.28) on square G and let $c'(0) \in W^{1,2}(G)$ then for every $G' \subset G$ there exist constant C, such that*

$$\begin{aligned} &\|\nabla c'\|_{L^2(L^2(G'))} + \|\nabla c'\|_{L^\infty(L^2(G'))} \\ &\qquad \leq C\left(\|\nabla c'(0)\|_2, \|c'\|_{L^2(W^{1,2}(G))}, \|\boldsymbol{v}'\|_{L^4(L^4(G))}, \|\boldsymbol{v}'\|_{L^2(W^{1,2}(G))}\right). \end{aligned} \tag{3.34}$$

Proof : Since any $\psi' \in L^2(0,T;W^{1,2}(G))$ with compact support in $G \times [0,T]$ is admissible test function in (3.27), we can choose $-\chi_{[0,t]} \mathrm{D}_s^{-h}\left(\eta^2 \mathrm{D}_s^h(c')\right)$ as test function (η is cutoff function), estimate the differences and obtain required estimate.

For $\epsilon' < \epsilon$ we choose cutoff function η to be $\mathcal{C}^\infty$ with compact support in G and to satisfy following conditions.

$$\begin{aligned} 0 \leq \eta(y) \leq 1 \qquad & \text{for all } y \in G \\ \eta(y) = 1 \qquad & \text{for all } y \in (-\epsilon', \epsilon') \times (-\epsilon', \epsilon') \end{aligned}$$

Using properties of differences we will estimate individual terms of (3.27) starting with the time derivative. We will assume that $h < \operatorname{dist}(\partial G, \operatorname{supp}(\eta))$ in order to $\mathrm{D}_s^{-h}\left(\eta^2 \mathrm{D}_s^h(c')\right)$ be well defined.

$$\begin{aligned} -\int_0^t \left\langle c'_{,t}; \mathrm{D}_s^{-h}\left(\eta^2 \mathrm{D}_s^h(c')\right)\right\rangle_G &= \int_0^t \left\langle \eta \frac{\mathrm{d}}{\mathrm{dt}} \mathrm{D}_s^h(c'); \eta \mathrm{D}_s^h(c')\right\rangle_G \\ &= \int_0^t \left\langle \frac{\mathrm{d}}{\mathrm{dt}}\left(\eta \mathrm{D}_s^h(c')\right) - \eta_{,t} \mathrm{D}_s^h(c'); \eta \mathrm{D}_s^h(c')\right\rangle_G \\ &\geq \frac{1}{2}\left\|\eta \mathrm{D}_s^h(c'(t))\right\|_2^2 - \frac{1}{2}\left\|\eta \mathrm{D}_s^h(c'(0))\right\|_2^2 - C\|\nabla c'\|_{L^2(L^2(G))}^2 \end{aligned} \tag{3.35}$$

Now the elliptic term.

$$
\begin{aligned}
-\int_0^t \int_G \alpha_{ij}(y)\frac{\partial c'}{\partial y_i}(y)\,\frac{\partial}{\partial y_j}\mathrm{D}_s^{-h}\left(\eta^2\mathrm{D}_s^h\left(c'\right)\right)(y)\,\mathrm{dy}& \\
= \int_0^t \int_G \mathrm{D}_s^h\left(\alpha_{ij}\frac{\partial c'}{\partial y_i}\right)(y)\frac{\partial}{\partial y_j}\eta^2\mathrm{D}_s^h\left(c'\right)(y)\,\mathrm{dy}& \\
= \int_0^t \int_G \left(\mathrm{D}_s^h\left(\alpha_{ij}\right)(y)\frac{\partial c'}{\partial y_i}(y) + \tau_h^s\left(\alpha_{ij}\right)(y)\frac{\partial}{\partial y_i}\mathrm{D}_s^h\left(c'\right)(y)\right)& \\
\left(2\eta(y)\frac{\partial \eta}{\partial y_j}(y)\mathrm{D}_s^h\left(c'\right)(y) + \eta^2(y)\frac{\partial}{\partial y_j}\mathrm{D}_s^h\left(c'\right)(y)\right)\mathrm{dy}& \quad (3.36)
\end{aligned}
$$

We split the estimate to four branches each for product of one term from first and one from the second bracket. We start with estimate of product of first terms from above. Since α_{ij} is Lipschitz continuous (lemma 3.3) the difference of α_{ij} is bounded.

$$
\begin{aligned}
\int_0^t \int_G \mathrm{D}_s^h\left(\alpha_{ij}\right)(y)\frac{\partial c'}{\partial y_i}(y)\,2\eta(y)\frac{\partial \eta}{\partial y_j}(y)\mathrm{D}_s^h\left(c'\right)(y)\mathrm{dy}& \\
\leq C\int_0^t \left\|\nabla c'\right\|_2 \left\|\mathrm{D}_s^h\left(c'\right)\right\|_2 \leq C\left\|\nabla c'\right\|_{L^2(L^2(G))}^2&
\end{aligned}
$$

Where constant C is depends only $\nabla\eta$ and derivatives of T. Now we estimate product of first term from first bracket and second term from second bracket from above. In order to estimate the term we will use Holder's and Young's inequality.

$$
\begin{aligned}
\int_0^t \int_G \mathrm{D}_s^h\left(\alpha_{ij}\right)(y)\frac{\partial c'}{\partial y_i}(y)\,\eta^2(y)\frac{\partial}{\partial y_j}\mathrm{D}_s^h\left(c'\right)(y)\,\mathrm{dy}& \\
\leq C\int_0^t \left\|\nabla c'\right\|_2 \left\|\eta\mathrm{D}_s^h\left(\nabla c'\right)\right\|_2 \leq C\left\|\nabla c'\right\|_{L^2(L^2(G))}\left\|\eta\mathrm{D}_s^h\left(\nabla c'\right)\right\|_{L^2(L^2(G))}& \\
\leq \frac{C}{\delta}\left\|\nabla c'\right\|_{L^2(L^2(G))}^2 + \delta\left\|\eta\mathrm{D}_s^h\left(\nabla c'\right)\right\|_{L^2(L^2(G))}^2&
\end{aligned}
$$

Where δ is arbitrary (will be chosen small). Next term to estimate is product of second term from first bracket and first term from second bracket.

$$
\begin{aligned}
\int_0^t \int_G \tau_h^s\left(\alpha_{ij}\right)(y)\frac{\partial}{\partial y_i}\mathrm{D}_s^h\left(c'\right)(y)\,2\eta(y)\frac{\partial \eta}{\partial y_j}(y)\mathrm{D}_s^h\left(c'\right)(y)\mathrm{dy}& \\
= C\int_0^t \left\|\eta\mathrm{D}_s^h\left(\nabla c'\right)\right\|_2 \left\|\mathrm{D}_s^h\left(c'\right)\right\|_2 \leq C\left\|\eta\nabla c'\right\|_{L^2(L^2(G))}\left\|\eta\mathrm{D}_s^h\left(\nabla c'\right)\right\|_{L^2(L^2(G))}& \\
\leq \frac{C}{\delta}\left\|\eta\nabla c'\right\|_{L^2(L^2(G))}^2 + \delta\left\|\eta\mathrm{D}_s^h\left(\nabla c'\right)\right\|_{L^2(L^2(G))}^2&
\end{aligned}
$$

We can estimate the last remaining term from bellow because α_{ij} is uniformly elliptic (Lemma 3.5).

$$
\int_0^t \int_G \tau_h^s\left(\alpha_{ij}\right)(y)\frac{\partial}{\partial y_i}\mathrm{D}_s^h\left(c'\right)(y)\,\eta^2(y)\frac{\partial}{\partial y_j}\mathrm{D}_s^h\left(c'\right)(y)\,\mathrm{dy} \geq \mu\int_0^t \left\|\eta\mathrm{D}_s^h\left(\nabla c'\right)\right\|_2^2
$$

Next to estimate is the convective term.

$$\int_0^t \int_G \frac{\partial c'}{\partial y_k}(y)\, \boldsymbol{v}'_k(y)\, \mathrm{D}_s^{-h}\left(\eta^2 \mathrm{D}_s^h(c')\right) \mathrm{dy}$$
$$= \int_0^t \int_G c'(y)\, \boldsymbol{v}'_k(y) \frac{\partial}{\partial y_k}\left(\mathrm{D}_s^{-h}\left(\eta^2 \mathrm{D}_s^h(c')\right)\right)(y)\mathrm{dy}$$
$$= \int_0^t \int_G \mathrm{D}_s^h\left(c'(y)\, \boldsymbol{v}'_k(y)\right) \frac{\partial}{\partial y_k}\left(\eta^2 \mathrm{D}_s^h(c')\right)(y)\mathrm{dy}$$
$$= \int_0^t \int_G \left(\mathrm{D}_s^h(c')(y)\, \boldsymbol{v}'_k(y) + \tau_h^s(c')(y)\, \mathrm{D}_s^h(\boldsymbol{v}'_k)(y)\right)$$
$$\left(2\eta \frac{\partial \eta}{\partial y_k} \mathrm{D}_s^h(c')(y) + \eta^2 \frac{\partial}{\partial y_k}\left(\mathrm{D}_s^h(c')\right)(y)\right) \mathrm{dy} \quad (3.37)$$

We estimate individual terms from the product starting with product of first terms.

$$\int_0^t \int_G \mathrm{D}_s^h(c')(y)\, \boldsymbol{v}'_k(y)\, 2\eta \frac{\partial \eta}{\partial y_k} \mathrm{D}_s^h(c')(y)\mathrm{dy} \le C \int_0^t \|\boldsymbol{v}'\|_4 \left\|\mathrm{D}_s^h(c')\right\|_2 \left\|\eta \mathrm{D}_s^h(c')\right\|_4$$

We use interpolation and Young's inequality and the fact that $\|\boldsymbol{v}'\|_{L^4(L^4(G))}$ is bounded by constant dependent only on data.

$$\le C \int_0^t \|\boldsymbol{v}'\|_4 \left\|\mathrm{D}_s^h(c')\right\|_2 \left\|\eta \mathrm{D}_s^h(c')\right\|_2^{\frac{1}{2}} \left\|\eta \mathrm{D}_s^h(c')\right\|_{1,2}^{\frac{1}{2}}$$
$$\le \int_0^t \left(C(\delta) \left\|\mathrm{D}_s^h(c')\right\|_2^2 + C(\delta) \|\boldsymbol{v}'\|_4^4 \left\|\eta \mathrm{D}_s^h(c')\right\|_2^2 + \delta \left\|\eta \mathrm{D}_s^h(c')\right\|_{1,2}^2\right)$$
$$\le \int_0^t C(\delta) \left(\left\|\mathrm{D}_s^h(c')\right\|_2^2 + C(\delta) \|\boldsymbol{v}'\|_4^4 \left\|\eta \mathrm{D}_s^h(c')\right\|_2^2 + \delta \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2^2\right)$$
$$\le C(\delta) \|\nabla c'\|_{L^2(L^2(G))}^2 + C(\delta) \int_0^t \|\boldsymbol{v}'\|_4^4 \left\|\eta \mathrm{D}_s^h(c')\right\|_2^2 + \delta \int_0^t \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2^2$$

Product of first term from first bracket and second term from second bracket can be estimated in the following way.

$$\begin{aligned}
\int_0^t \int_G \mathrm{D}_s^h(c')(y)\,\boldsymbol{v}_k'(y)\,\eta^2 \frac{\partial}{\partial y_k}\left(\mathrm{D}_s^h(c')\right)(y)\mathrm{d}y \;&\leq \int_0^t \left\|\eta \mathrm{D}_s^h(c')\right\|_4 \|\boldsymbol{v}'\|_4 \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2 \\
&\leq \int_0^t \left\|\eta \mathrm{D}_s^h(c')\right\|_2^{\frac{1}{2}} \left\|\eta \mathrm{D}_s^h(c')\right\|_{1,2}^{\frac{1}{2}} \|\boldsymbol{v}'\|_4 \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2 \\
&\leq C\int_0^t \left\|\eta \mathrm{D}_s^h(c')\right\|_2^{\frac{1}{2}} \left\|\eta \mathrm{D}_s^h(c')\right\|_{1,2}^{\frac{1}{2}} \|\boldsymbol{v}'\|_4 \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2 \\
&\leq C\int_0^t \left\|\eta \mathrm{D}_s^h(c')\right\|_2^{\frac{1}{2}} \|\boldsymbol{v}'\|_4 \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2^{\frac{3}{2}} \\
&\quad + C\int_0^t \left\|\eta \mathrm{D}_s^h(c')\right\|_2^{\frac{1}{2}} \left\|\mathrm{D}_s^h(c')\right\|_2^{\frac{1}{2}} \|\boldsymbol{v}'\|_4 \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2 \\
&\leq C(\delta)\,\|c'\|_{L^2(W^{1,2}(G))} + C(\delta)\int_0^t \|\boldsymbol{v}'\|_4^4 \left\|\eta \mathrm{D}_s^h(c')\right\|_2^2 + \delta\int_0^t \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2^2
\end{aligned}$$

We estimate product of second term from first bracket and first term from second bracket as below.

$$\begin{aligned}
\int_0^t \int_G \tau_h^s(c')(y)\,\mathrm{D}_s^h(\boldsymbol{v}_k')(y)\,2\eta\frac{\partial \eta}{\partial y_k}\mathrm{D}_s^h(c')(y)\mathrm{d}y &\leq C\int_0^t \left\|\mathrm{D}_s^h(\boldsymbol{v}')\right\|_2 \left\|\eta \mathrm{D}_s^h(c')\right\|_2 \\
&\leq C\,\|\nabla \boldsymbol{v}'\|_{L^2(L^2(G))}^2 + C\int_0^t \left\|\eta \mathrm{D}_s^h(c')\right\|_2^2
\end{aligned}$$

And finally we estimate last remaining term as follows.

$$\begin{aligned}
\int_0^t \int_G \tau_h^s(c')(y)\,\mathrm{D}_s^h(\boldsymbol{v}_k')(y)\,\eta^2\frac{\partial}{\partial y_k}\left(\mathrm{D}_s^h(c')\right)(y)\mathrm{d}y &\leq \int_0^t \left\|\mathrm{D}_s^h(\boldsymbol{v}')\right\|_2 \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2 \\
&\leq C(\delta)\,\|\nabla \boldsymbol{v}'\|_{L^2(L^2(G))}^2 + \delta\int_0^t \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2^2
\end{aligned}$$

Putting all the above estimate together gives us (3.38).

$$\begin{aligned}
\frac{1}{2}\left\|\eta \mathrm{D}_s^h(c'(t))\right\|_2^2 + k\mu\int_0^t \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2^2 &\leq 4\delta\left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_{L^2(L^2(G))}^2 + \frac{1}{2}\left\|\eta \mathrm{D}_s^h(c'(0))\right\|_2^2 \\
+ C(\delta)\,\|\nabla c'\|_{L^2(L^2(G))}^2 + C\int_0^t &\left(1+\|\boldsymbol{v}'\|_4^4\right)\left\|\eta \mathrm{D}_s^h(c)'\right\|_2^2 + C(\delta)\,\|\nabla \boldsymbol{v}'\|_{L^2(L^2(G))}^2 \qquad (3.38)
\end{aligned}$$

We choose $\delta = \frac{k\mu}{8}$ and absorb second order term in left hand side to get

$$\begin{aligned}
\frac{1}{2}\left\|\eta \mathrm{D}_s^h(c'(t))\right\|_2^2 + \frac{k\mu}{2}\int_0^t \left\|\eta \mathrm{D}_s^h(\nabla c')\right\|_2^2 &\leq \frac{1}{2}\|\nabla c'(0)\|_2^2 + \frac{C}{\delta}\|\nabla c'\|_{L^2(L^2(G))}^2 \\
&+ C\int_0^t \left(1+\|\boldsymbol{v}'\|_4^4\right)\left\|\eta \mathrm{D}_s^h(c)'\right\|_2^2 + \frac{C}{\delta}\|\nabla \boldsymbol{v}'\|_{L^2(L^2(G))}^2. \qquad (3.39)
\end{aligned}$$

Since $\|\nabla c'\|^2_{L^2(L^2(G))}$, $\|\boldsymbol{v}'\|^4_4$ and $\|\nabla \boldsymbol{v}'\|^2_{L^2(L^2(G))}$ are integrable on $[0,T]$ and $\|\nabla c'(0)\|^2_2$ is finite, we can use Gronwall's inequality to estimate $\left\|\eta \mathrm{D}^h_s\left(c'(t)\right)\right\|^2_2$.

$$\left\|\eta \mathrm{D}^h_s\left(c'(t)\right)\right\|^2_2 \leq C\left(\|\nabla c'(0)\|_2, \|c'\|_{L^2(W^{1,2}(G))}, \|\boldsymbol{v}'\|_{L^4(L^4(G))}, \|\boldsymbol{v}'\|_{L^2(W^{1,2}(G))}\right) \tag{3.40}$$

Estimate on $\int_0^T \left\|\eta \mathrm{D}^h_s\left(\nabla c'\right)\right\|^2_2$ easily follows from (3.40) and (3.39). We so far obtained estimates on differences of concentration, since these are uniform with respect to h, we can use lemma 2.5 to obtain the same estimates on corresponding partial derivatives on set $(-\epsilon'', \epsilon'') \times (-\epsilon'', \epsilon'') \subset \{x \in G | \eta(x) = 1\} \subset G$, where $\epsilon'' < \epsilon'$ can be arbitrary. Since ϵ' can be chosen arbitrarily close to ϵ and ϵ'' can be chosen arbitrarily close to ϵ' we can for every $G' \subset G$ for which $\bar{G}' \subset G$ choose ϵ' and ϵ'' such that $G' \subset (-\epsilon'', \epsilon'')$ and therefore obtain the statement of the theorem.

□

Lemma 3.8 *For every $x_0 \in \overline{\Omega}$ there exists U open neighborhood of x_0 such that*

$$c \in L^2\left(0,T; W^{2,2}_{loc}\left(U_{x_0}\right)\right) \quad c \in L^\infty\left(0,T; W^{1,2}_{loc}\left(U_{x_0}\right)\right)$$

Proof : Since for x_0 in Ω is much simpler and can be don by simplifying the proof for x_0 on the boundary of Ω, we restrict ourselves to the case $x_0 \in \partial\Omega$. For x_0 we find U as in section 3.3 and transform the equation for concentration to the set G (same notation as in section 3.3). From lemma 3.7 follows c' (defined by 3.10) is in spaces $L^2\left(0,T; W^{2,2}_{loc}\left(G^+\right)\right)$ and $L^\infty\left(0,T; W^{1,2}_{loc}\left(G^+\right)\right)$, result follows from the fact that T is $\mathcal{C}^2$ diffeomorphism and definition of c'.

□

Theorem 3.9 *Let $\boldsymbol{v}$, p and c be weak solution of system* (2.3) *in the sense of definition 2.1 and let initial condition c_0 be in space $W^{1,2}(\Omega)$, then*

$$c \in L^2\left(0,T; W^{2,2}(\Omega)\right) \cap L^\infty\left(0,T; W^{1,2}(\Omega)\right)$$

Proof : For $x \in \overline{\Omega}$ arbitrary we find U_x open neighborhood of x from Lemma 3.8. Since $\{U_x | x \in \overline{\Omega}\}$ is open cover of $\overline{\Omega}$ and $\overline{\Omega}$ is compact we can find $\mathcal{A} \subset \{U_x | x \in \overline{\Omega}\}$, which is finite and covers $\overline{\Omega}$. Let $\{\eta_U | U \in \mathcal{A}\}$ be smooth partition of unity subordinate to $\mathcal{A}$.

$$\begin{aligned} \operatorname{supp}(\eta_U) &\subset U \\ \sum_{U \in \mathcal{A}} \eta_U &= 1 \end{aligned}$$

Because of Lemma 3.8 functions $\eta_U c$ are in spaces $L^2\left(0,T; W^{2,2}(\Omega)\right)$ and $L^\infty\left(0,T; W^{1,2}(\Omega)\right)$, therefore

$$c = \sum_{U \in \mathcal{A}} \eta_U c$$

is in the desired spaces.

□

3.7 Estimates on second derivatives of velocity

Because of technical problems (see proof of Lemma 3.10), which arise from viscosity being non-constant, we cannot estimate second derivatives of velocity as easily as second derivatives of concentration. We will approximate the velocity, estimate the approximate velocity, show the convergence of the approximations and transfer the estimates on the original velocity.

We introduce approximation of the balance of momentum. We approximate c with sequence of Lipschitz continuous functions c_n, such that

$$\begin{aligned} 0 \le c_n(x,t) \le 1 \qquad & \text{for almost all } (x,t) \in Q, \\ c_n \to c \qquad & \text{in space } L^4\left(0,T;W^{1,4}\left(\Omega\right)\right). \end{aligned}$$

Since $c \in L^4\left(0,T;W^{1,4}\left(\Omega\right)\right)$ (Theorem 3.9 and interpolation inequality) such a sequence can be constructed by convolution with standard mollifier (for details see [5]).

We define approximation $\boldsymbol{v}_n$ of $\boldsymbol{v}$ as weak solution to the system

$$\begin{aligned} \frac{\partial \boldsymbol{v}_n}{\partial t} - \operatorname{div}\left(\nu\left(c_n\right)\mathbf{D}\left(\boldsymbol{v}_n\right)\right) + \operatorname{div}\left(\boldsymbol{v}_n \otimes \boldsymbol{v}_n\right) &= -\nabla p_n + \boldsymbol{f} \\ \operatorname{div} \boldsymbol{v}_n &= 0 \end{aligned} \tag{3.41}$$

with Navier boundary conditions and initial condition $\boldsymbol{v}_0$. Theory for generalized Navier-Stokes (see [3]) equation gives us estimate (3.42), where constant C does not depend on n.

$$\|\boldsymbol{v}_n\|_{L^\infty(L^2)} + \|\boldsymbol{v}_n\|_{L^2(W^{1,2})} + \|p_n\|_{L^2(L^2)} \le C \tag{3.42}$$

We transform and extend c_n, p_n and $\boldsymbol{v}_n$ in the same way as we transformed the original quantities (section 3.3). We denote the transformed quantities c_n', p_n' and $\boldsymbol{v}_n'$. Since $\boldsymbol{v}_n$ and p_n satisfies (3.41), it is easy to see that p_n' and $\boldsymbol{v}_n'$ satisfies (3.28) with c' replaced with c_n'. We estimate $\boldsymbol{v}_n'$ in the following lemma.

Lemma 3.10 *Let $\boldsymbol{v}'$ and p' satisfies* (3.28) *for some function c' ($0 < c' < 1$) and let $\boldsymbol{v}'(0) \in W^{1,2}\left(G\right)$ then for every $G' \subset G$ such that $\overline{G'} \subset G$ following statements hold.*

1. *If c' is Lipschitz continuous then there exist constant C such that*

$$\left\|\nabla^2 \boldsymbol{v}'\right\|_{L^2(L^2(G'))} + \left\|\nabla \boldsymbol{v}'\right\|_{L^\infty(L^2(G'))} \le$$
$$C\left(\nu, \|c'\|_{1,\infty,Q}, \|\nabla \boldsymbol{v}'(0)\|_{2,G}, \|\nabla c'\|_{L^2(L^2(G))}, \|\boldsymbol{v}'\|_{L^4(L^4(G))}, \|p'\|_{L^2(L^2(G))}, \|\boldsymbol{v}'\|_{L^2(W^{1,2}(G))}\right) \tag{3.43}$$

2. *Let $\boldsymbol{v}'$ belongs to the space $L^2\left(0,T;W^{2,2}\left(G\right)\right) \cap L^\infty\left(0,T;W^{1,2}\left(G\right)\right)$ and let c' belongs to the space $L^4\left(0,T;W^{1,4}\left(G\right)\right)$, then there exist constant C such that*

$$\left\|\nabla^2 \boldsymbol{v}'\right\|_{L^2(L^2(G'))} + \left\|\nabla \boldsymbol{v}'\right\|_{L^\infty(L^2(G'))} \le$$
$$C\left(\nu, \|\boldsymbol{v}'(0)\|_{1,2,G}, \|\boldsymbol{v}'\|_{L^4(L^4(G))}, \|\boldsymbol{v}'\|_{L^2(W^{1,2}(G))}, \|c'\|_{L^4(W^{1,4}(G))}, \|p'\|_{L^2(L^2(G))}\right) \tag{3.44}$$

.

Proof : Since proof for both parts is very similar, we will carry out only one proof and in the parts when proofs of 1 and 2 differ we will split the proof in two branches. We will derive the estimates by using second difference of $\boldsymbol{v}'$ (similarly as in theorem 3.7) as test function, but we have to be careful since equation (3.28) is satisfied only for functions which has zero trace of second component on G^0. However if we choose the cutoff function η suitably we can use $-\chi_{[0,t]}\mathrm{D}_s^{-h}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right)$ as test function in 3.28.

For $\epsilon' < \epsilon$ we choose $\mathcal{C}^\infty$ cutoff function η such that

$$\begin{aligned} \operatorname{supp}(\eta) &\subset G && && (3.45)\\ 0 \le \eta &\le 1 && \text{on } G && (3.46)\\ \eta &= 1 && \text{on } (-\epsilon',\epsilon')\times(-\epsilon',\epsilon') && (3.47)\end{aligned}$$

and such that there exist $\lambda > 0$ such that for every $x_1 \in (-\epsilon,\epsilon)$ function $x_2 \to \eta(x_1,x_2)$ is constant on $(-\lambda,\lambda)$. If η is defined as above and $h < \min\left(\lambda, \operatorname{dist}\left(\partial G, \operatorname{supp}(\eta)\right)\right)$, then $\mathrm{D}_2^h\left(\eta^2\mathrm{D}_2^h\left(\boldsymbol{v}_2'\right)\right)$ has zero trace on G^0, because $\boldsymbol{v}'$ is odd in second variable.

$$\begin{aligned} h^2\mathrm{D}_2^{-h}\left(\eta^2\mathrm{D}_2^h\left(\boldsymbol{v}_2'\right)\right)(x_1,0) &= \eta^2(x_1,-h)\mathrm{D}_2^h\left(\boldsymbol{v}_2'\right)(x_1,-h) - \eta^2(x_1,0)\mathrm{D}_2^h\left(\boldsymbol{v}_2'\right)(x_1,0)\\ &= \eta^2(x_1,0)\left(\boldsymbol{v}_2'(x_1,-h) - 2\boldsymbol{v}_2'(x_1,0) + \boldsymbol{v}_2'(x_1,h)\right) = 2\eta^2(x_1,0)\boldsymbol{v}_2'(x_1,0) = 0 \quad (3.48)\end{aligned}$$

$\mathrm{D}_1^h\left(\eta\mathrm{D}_1^h\left(\boldsymbol{v}_2'\right)\right)$ is zero on G^0, since $\boldsymbol{v}_2' = 0$ on G^0. We estimate individual terms of 3.28 starting with the time derivative :

$$\begin{aligned} -\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}'; [\nabla T]^T\nabla T\chi_{[0,t]}\mathrm{D}_s^{-h}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right)\right\rangle_G &= \int_0^t \left\langle \mathrm{D}_s^h\left([\nabla T]^T\nabla T\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}'\right); \eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right\rangle_G\\ = \int_0^t \left\langle \mathrm{D}_s^h\left([\nabla T]^T\nabla T\right)\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}' + \tau_h^s\left([\nabla T]^T\nabla T\right)\mathrm{D}_s^h\left(\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}'\right); \eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right\rangle_G\\ = \int_0^t \left\langle \mathrm{D}_s^h\left([\nabla T]^T\nabla T\right)\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}'; \eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right\rangle_G\\ + \int_0^t \left\langle \frac{\mathrm{d}}{\mathrm{dt}}\left(\eta\tau_h^s\left(\nabla T\right)\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right); \tau_h^s\left(\nabla T\right)\eta\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right\rangle_G\\ \ge \frac{1}{2}\left\|\tau_h^s\left(\nabla T\right)\eta\mathrm{D}_s^h\left(\boldsymbol{v}'(t)\right)\right\|_2^2 - \frac{1}{2}\left\|\tau_h^s\left(\nabla T\right)\eta\mathrm{D}_s^h\left(\boldsymbol{v}'(0)\right)\right\|_2^2\\ - \int_0^t \left\|\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}'\right\|_{-1,2}\left\|\mathrm{D}_s^h\left([\nabla T]^T\nabla T\right)\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right\|_{1,2}\\ \ge \frac{c}{2}\left\|\eta\mathrm{D}_s^h\left(\boldsymbol{v}'(t)\right)\right\|_2^2 - \frac{C}{2}\left\|\eta\mathrm{D}_s^h\left(\boldsymbol{v}'(0)\right)\right\|_2^2 - C\int_0^t \left\|\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}'\right\|_{-1,2}\left\|\eta\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right\|_{1,2}\\ \ge \frac{c}{2}\left\|\eta\mathrm{D}_s^h\left(\boldsymbol{v}'(t)\right)\right\|_2^2 - \frac{C}{2}\left\|\eta\mathrm{D}_s^h\left(\boldsymbol{v}'(0)\right)\right\|_2^2 - C\left\|\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}'\right\|_{L^2(W^{-1,2}(G))}\left\|\eta\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right\|_{L^2(W^{1,2}(G))}.\end{aligned}$$

Since the splitting occurs in elliptic term we proceed with estimates on the first part of the convective term convective term :

$$
\begin{aligned}
\int_0^T \int_G \pi_{kn}(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y)\, \boldsymbol{v}'_l(y)\, \chi_{[0,t]} \mathrm{D}_s^{-h}\left(\eta^2 \mathrm{D}_s^h(\boldsymbol{v}'_n)\right)(y)\, \mathrm{dy} \\
\leq \int_0^t \int_G \mathrm{D}_s^h \left(\pi_{kn}(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y)\, \boldsymbol{v}'_l(y)\right) \eta^2 \mathrm{D}_s^h(\boldsymbol{v}'_n)(y)\, \mathrm{dy} \\
\leq \int_0^t \int_G \mathrm{D}_s^h(\pi_{kn})(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y)\, \boldsymbol{v}'_l(y)\, \eta^2 \mathrm{D}_s^h(\boldsymbol{v}'_n)(y)\, \mathrm{dy} \\
+ \int_0^t \int_G \tau_h^s(\pi_{kn})(y) \mathrm{D}_s^h\left(\frac{\partial \boldsymbol{v}'_k}{\partial y_l}\right)(y)\, \tau_h^s(\boldsymbol{v}'_l)(y)\, \eta^2 \mathrm{D}_s^h(\boldsymbol{v}'_n)(y)\, \mathrm{dy} \\
+ \int_0^t \int_G \tau_h^s(\pi_{kn})(y) \frac{\partial \boldsymbol{v}'_k}{\partial y_l}(y)\, \mathrm{D}_s^h(\boldsymbol{v}'_l)(y)\, \eta^2 \mathrm{D}_s^h(\boldsymbol{v}'_n)(y)\, \mathrm{dy} \\
\leq C \int_0^t \|\nabla \boldsymbol{v}'\|_2 \|\boldsymbol{v}'\|_4 \left\|\eta \mathrm{D}_s^h(\boldsymbol{v}'_n)\right\|_4 + C \int_0^t \|\nabla \boldsymbol{v}'\|_2 \left\|\eta \mathrm{D}_s^h(\boldsymbol{v}')\right\|_4^2 \\
+ C \int_0^t \left\|\eta \mathrm{D}_s^h(\nabla \boldsymbol{v}')\right\|_2 \|\tau_h^s(\boldsymbol{v}'_l)\|_4 \left\|\eta \mathrm{D}_s^h(\boldsymbol{v}')\right\|_4 \\
\leq C \int_0^t \|\nabla \boldsymbol{v}'\|_2 \|\boldsymbol{v}'\|_4 \left\|\eta \mathrm{D}_s^h(\boldsymbol{v}'_n)\right\|_2^{\frac{1}{2}} \left\|\eta \mathrm{D}_s^h(\boldsymbol{v}'_n)\right\|_{1,2}^{\frac{1}{2}} \\
+ C \int_0^t \|\nabla \boldsymbol{v}'\|_2 \left\|\eta \mathrm{D}_s^h(\boldsymbol{v}')\right\|_{1,2} \left\|\eta \mathrm{D}_s^h(\boldsymbol{v}')\right\|_2 \\
+ C \int_0^t \left\|\eta \mathrm{D}_s^h(\nabla \boldsymbol{v}')\right\|_2 \|\boldsymbol{v}'\|_4 \left\|\eta \mathrm{D}_s^h(\boldsymbol{v}')\right\|_2^{\frac{1}{2}} \left\|\eta \mathrm{D}_s^h(\boldsymbol{v}')\right\|_{1,2}^{\frac{1}{2}} \\
\leq C(\delta) \int_0^t \left(1 + \|\boldsymbol{v}'\|_4^4 + \|\nabla \boldsymbol{v}'\|_2^2\right) \left\|\eta \mathrm{D}_s^h(\boldsymbol{v}'_n)\right\|_2^2 \\
+ C(\delta) \int_0^t \|\nabla \boldsymbol{v}'\|_2 + \delta \int_0^t \left\|\eta \mathrm{D}_s^h(\nabla \boldsymbol{v}')\right\|_2^2 .
\end{aligned}
$$

Now we estimate the second part of the convective term :

$$
\begin{aligned}
\int_0^T \int_G \lambda_{lkn}(y) \boldsymbol{v}'_l(y)\, \boldsymbol{v}'_k(y)\, \chi_{[0,t]} \mathrm{D}_s^{-h}\left(\eta^2 \mathrm{D}_s^h(\boldsymbol{v}'_n)\right)(y)\, \mathrm{dy} \\
\leq C \int_0^t \|\boldsymbol{v}'\|_4^2 \left\|\mathrm{D}_s^{-h}\left(\eta^2 \mathrm{D}_s^h(\boldsymbol{v}'_n)\right)\right\|_2 \leq C(\delta) \int_0^t \|\boldsymbol{v}'\|_4^4 + \delta \int_0^t \left\|\nabla\left(\eta^2 \mathrm{D}_s^h(\boldsymbol{v}'_n)\right)\right\|_2^2 \\
\leq C(\delta) \int_0^t \|\boldsymbol{v}'\|_4^4 + \delta \int_0^t \left\|\left(\eta \mathrm{D}_s^h(\nabla \boldsymbol{v}'_n)\right)\right\|_2^2 + C(\delta) \int_0^t \left\|\eta \mathrm{D}_s^h(\boldsymbol{v}')\right\|_2^2 .
\end{aligned}
$$

Estimate of pressure term is not difficult, since we have chosen T to preserve volume and therefore we can use the fact that $\operatorname{div} \boldsymbol{v}' = 0$.

$$
\begin{aligned}
&\int_0^T \int_G p'(y) \frac{\partial}{\partial y_i} \left(\chi_{[0,t]} \mathrm{D}_s^{-h} \left(\eta^2 \mathrm{D}_s^h \left(\boldsymbol{v}_i' \right) \right) \right) (y) \,\mathrm{dy} \\
&\qquad \le \int_0^t \int_G p'(y) \mathrm{D}_s^{-h} \left(2 \frac{\partial \eta}{\partial y_i} \eta \mathrm{D}_s^h \left(\boldsymbol{v}_i' \right) + \eta^2 \mathrm{D}_s^h \left(\frac{\partial \boldsymbol{v}_i'}{\partial y_i} \right) \right) (y) \,\mathrm{dy} \\
&\qquad \le \int_0^t \int_G p'(y) \left(\mathrm{D}_s^{-h} \left(\frac{\partial \eta}{\partial y_i} \right) (y) \eta(y) \mathrm{D}_s^h \left(\boldsymbol{v}_i' \right) (y) \right. \\
&\qquad\qquad \left. + \tau_h^s \left(\frac{\partial \eta}{\partial y_i} \right) (y) \mathrm{D}_s^{-h} \left(\eta \mathrm{D}_s^h \left(\boldsymbol{v}_i' \right) \right) (y) \right) \mathrm{dy} \\
&\qquad \le C \int_0^t \| p' \|_2 \left(\left\| \eta \mathrm{D}_s^h \left(\boldsymbol{v}' \right) \right\|_2 + \left\| \mathrm{D}_s^{-h} \left(\eta \mathrm{D}_s^h \left(\boldsymbol{v}_i' \right) \right) \right\|_2 \right) \\
&\qquad \le C(\delta) \int_0^t \left(\| p' \|_2^2 + \| \boldsymbol{v}' \|_{1,2}^2 \right) + \delta \int_0^t \left\| \eta \mathrm{D}_s^{-h} \left(\nabla \boldsymbol{v}' \right) \right\|_2^2
\end{aligned}
$$

Before we go to elliptic term, it remains to estimate the term with external force :

$$
\begin{aligned}
&\int_0^T \int_G \pi_{kl}(y) \boldsymbol{f}_k'(y) \chi_{[0,t]} \mathrm{D}_s^{-h} \left(\eta^2 \mathrm{D}_s^h \left(\boldsymbol{v}_l' \right) \right) (y) \mathrm{dy} \\
&\qquad \le C(\delta) \int_0^t \| \boldsymbol{f}' \|_2^2 + \delta \int_0^t \left\| \mathrm{D}_s^{-h} \left(\eta^2 \mathrm{D}_s^h \left(\boldsymbol{v}' \right) \right) \right\|_2^2 \\
&\qquad \le C(\delta) \int_0^t \| \boldsymbol{f}' \|_2^2 + \delta \int_0^t \left\| \eta \mathrm{D}_s^h \left(\nabla \boldsymbol{v}' \right) \right\|_2^2 + C(\delta) \int_0^t \| \boldsymbol{v}' \|_{1,2}^2 .
\end{aligned}
$$

Finally we estimate terms which were produced by transformation of elliptic term as follows :

$$
\begin{aligned}
&\frac{1}{4} \int_0^T \int_G \nu \left(c'(y) \right) \vartheta_{lm}(y) \boldsymbol{v}_l'(y) \chi_{[0,t]} \mathrm{D}_s^{-h} \left(\eta^2 \mathrm{D}_s^h \left(\boldsymbol{v}_m' \right) \right) (y) \mathrm{dy} \\
&\qquad \le C \int_0^t \| \boldsymbol{v}' \|_2 \left\| \mathrm{D}_s^{-h} \left(\eta^2 \mathrm{D}_s^h \left(\boldsymbol{v}' \right) \right) \right\|_2 \le C(\delta) \int_0^t \| \boldsymbol{v}' \|_2^2 + \delta \int_0^t \left\| \nabla \left(\eta^2 \mathrm{D}_s^h \left(\boldsymbol{v}' \right) \right) \right\|_2^2 \\
&\qquad \le C(\delta) \int_0^t \| \boldsymbol{v}' \|_{1,2}^2 + \delta \int_0^t \left\| \left(\eta \mathrm{D}_s^h \left(\nabla \boldsymbol{v}' \right) \right) \right\|_2^2 .
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{4} \int_0^T \int_G \nu \left(c'(y) \right) \sigma_{mkl}(y) \frac{\partial \boldsymbol{v}_k'}{\partial y_l} (y) \, \chi_{[0,t]} \mathrm{D}_s^{-h} \left(\eta^2 \mathrm{D}_s^h \left(\boldsymbol{v}_m' \right) \right) (y) \mathrm{dy} \le C \int_0^t \| \boldsymbol{v}' \|_{1,2} \left\| \mathrm{D}_s^{-h} \left(\eta^2 \mathrm{D}_s^h \left(\boldsymbol{v}_m' \right) \right) \right\|_2 \\
&\le C(\delta) \int_0^t \| \boldsymbol{v}' \|_{1,2}^2 + \delta \int_0^t \left\| \mathrm{D}_s^{-h} \left(\eta^2 \mathrm{D}_s^h \left(\boldsymbol{v}_m' \right) \right) \right\|_2^2 \le C(\delta) \int_0^t \| \boldsymbol{v}' \|_{1,2}^2 + \delta \int_0^t \left\| \left(\eta \mathrm{D}_s^h \left(\nabla \boldsymbol{v}' \right) \right) \right\|_2^2
\end{aligned}
$$

For the next term we use lemma 3.4 to handle $\sigma_{lmn}\boldsymbol{v}'_l$ and get

$$\begin{aligned}
&\frac{1}{4}\int_0^T\int_G \nu\left(c'(y)\right)\sigma_{lmn}(y)\boldsymbol{v}'_l(y)\frac{\partial}{\partial y_n}\chi_{[0,t]}\mathrm{D}_s^{-h}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'_m\right)\right)(y)\mathrm{dy}\\
&\qquad\le\frac{1}{4}\int_0^t\int_G \mathrm{D}_s^h\left(\nu\left(c'\right)\sigma_{lmn}\boldsymbol{v}'_l\right)(y)\frac{\partial}{\partial y_n}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'_m\right)(y)\right)\mathrm{dy}\\
&\qquad\le\frac{1}{4}\int_0^t\int_G \mathrm{D}_s^h\left(\nu\left(c'\right)\right)(y)\sigma_{lmn}(y)\boldsymbol{v}'_l(y)\frac{\partial}{\partial y_n}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'_m\right)(y)\right)\mathrm{dy}\\
&\qquad+\frac{1}{4}\int_0^t\int_G \tau_h^s\left(\nu\left(c'\right)\right)(y)\mathrm{D}_s^h\left(\sigma_{lmn}(y)\boldsymbol{v}'_l(y)\right)\frac{\partial}{\partial y_n}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'_m\right)(y)\right)\mathrm{dy}\\
&\qquad\le C\int_0^t\left\|\nabla c'\right\|_4\left\|\sigma_{lmn}(y)\boldsymbol{v}'_l(y)\right\|_4\left\|\nabla\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right)\right\|_2\\
&\qquad\quad+C\int_0^t\left\|\nabla\left(\sigma_{lmn}\boldsymbol{v}'_l\right)\right\|_2\left\|\nabla\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'_m\right)\right)\right\|_2\\
&\qquad\le C(\delta)\int_0^t\left(\left\|\nabla c'\right\|_4^4+\left\|\boldsymbol{v}'\right\|_4^4+\left\|\boldsymbol{v}'\right\|_{1,2}^2\right)+\delta\int_0^t\left\|\eta\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right\|_2
\end{aligned}$$

Finally it remains to estimate the last term produced by transformation of the elliptic term.

$$\begin{aligned}
&\frac{1}{4}\int_0^T\int_G \nu\left(c'(y)\right)\omega_{mnkl}(y)\frac{\partial\boldsymbol{v}'_k}{\partial y_l}(y)\frac{\partial}{\partial y_n}\left(\chi_{[0,t]}\mathrm{D}_s^{-h}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'_m\right)\right)\right)(y)\,\mathrm{dy}\\
&\quad=\frac{1}{4}\int_0^t\int_G \mathrm{D}_s^h\left(\nu\left(c'(y)\right)\omega_{mnkl}(y)\frac{\partial\boldsymbol{v}'_k}{\partial y_l}\right)(y)\frac{\partial}{\partial y_n}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'_m\right)\right)(y)\mathrm{dy}\\
&\quad=\frac{1}{4}\int_0^t\int_G \mathrm{D}_s^h\left(\nu\left(c'(y)\right)\right)\tau_h^s\left(\omega_{mnkl}\right)(y)\frac{\partial\boldsymbol{v}'_k}{\partial y_l}(y)\frac{\partial}{\partial y_n}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'_m\right)\right)(y)\mathrm{dy}\\
&\quad+\frac{1}{4}\int_0^t\int_G \tau_h^s\left(\nu\left(c'(y)\right)\right)\mathrm{D}_s^h\left(\omega_{mnkl}\right)(y)\tau_h^s\left(\frac{\partial\boldsymbol{v}'_k}{\partial y_l}\right)(y)\frac{\partial}{\partial y_n}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'_m\right)(y)\right)\mathrm{dy}\\
&\qquad+\frac{1}{4}\int_0^t\int_G \nu\left(c'(y)\right)\omega_{mnkl}(y)\frac{\partial\mathrm{D}_s^h\left(\boldsymbol{v}'_k\right)}{\partial y_l}(y)\frac{\partial}{\partial y_n}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'_m\right)(y)\right)\mathrm{dy}
\end{aligned}\tag{3.49}$$

Since the splitting of the proof occurs in the first term we at first estimate second and third term of the last expression. We will estimate the second term from (3.49) as follows :

$$\begin{aligned}
&\frac{1}{4}\int_0^t\int_G \tau_h^s\left(\nu\left(c'(y)\right)\right)\mathrm{D}_s^h\left(\omega_{mnkl}\right)(y)\tau_h^s\left(\frac{\partial\boldsymbol{v}'_k}{\partial y_l}\right)(y)\frac{\partial}{\partial y_n}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'_m\right)(y)\right)\mathrm{dy}\\
&\qquad\le C\int_0^t\left\|\nabla\boldsymbol{v}'\right\|_2\left\|\nabla\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right)\right\|_2\le C(\delta)\int_0^t\left\|\boldsymbol{v}'\right\|_{1,2}^2+\delta\int_0^t\left\|\eta\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right\|_2^2.
\end{aligned}$$

We will estimate the third term from bellow in the following way :

$$
\begin{aligned}
&\frac{1}{4}\int_0^t\int_G \nu\left(c'(y)\right)\omega_{mnkl}(y)\frac{\partial \mathrm{D}_s^h\left(\boldsymbol{v}_k'\right)}{\partial y_l}(y)\frac{\partial}{\partial y_n}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}_m'\right)(y)\right)\mathrm{dy}\\
&\geq \frac{1}{4}\int_0^t\int_G \nu\left(c'(y)\right)\omega_{mnkl}(y)\frac{\partial \mathrm{D}_s^h\left(\boldsymbol{v}_k'\right)}{\partial y_l}(y)\eta\frac{\partial \eta\mathrm{D}_s^h\left(\boldsymbol{v}_m'\right)}{\partial y_n}(y)\mathrm{dy}-C\int_0^t\left\|\eta\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right\|_2\left\|\boldsymbol{v}'\right\|_{1,2}\\
&\qquad\qquad\geq \frac{1}{4}\int_0^t\int_G \nu\left(c'(y)\right)\omega_{mnkl}(y)\frac{\partial \eta\mathrm{D}_s^h\left(\boldsymbol{v}_k'\right)}{\partial y_l}(y)\frac{\partial \eta\mathrm{D}_s^h\left(\boldsymbol{v}_m'\right)}{\partial y_n}(y)\mathrm{dy}\\
&\qquad\qquad\qquad\qquad -\delta\int_0^t\left\|\eta\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right\|_2^2-C(\delta)\int_0^t\left\|\boldsymbol{v}'\right\|_{1,2}^2.
\end{aligned}
$$

At this point we use 3.6 to estimate the first term from bellow and get

$$
\begin{aligned}
&\frac{\kappa}{2C_K}\int_0^t\int_G\left|\nabla\left(\eta\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right)\right|^2-\delta\int_0^t\left\|\eta\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right\|_2^2-C(\delta)\int_0^t\left\|\boldsymbol{v}'\right\|_{1,2}^2\\
&\qquad\qquad\geq\frac{\kappa}{2C_K}\int_0^t\int_G\left|\eta\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right|^2-\delta\int_0^t\left\|\eta\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right\|_2^2-C(\delta)\int_0^t\left\|\boldsymbol{v}'\right\|_{1,2}^2.
\end{aligned}
$$

Now we split the proof in two branches and estimate first term from (3.49).

1. Since c' is Lipschitz continuous, $\mathrm{D}_s^h\left(\nu\left(c'(y)\right)\right)$ is bounded and therefore

$$
\begin{aligned}
&\frac{1}{4}\int_0^t\int_G \mathrm{D}_s^h\left(\nu\left(c'(y)\right)\right)\tau_h^s\left(\omega_{mnkl}\right)(y)\frac{\partial\boldsymbol{v}_k'}{\partial y_l}(y)\frac{\partial}{\partial y_n}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}_m'\right)\right)(y)\mathrm{dy}\\
&\qquad\qquad\leq C\left\|c'\right\|_{1,\infty,Q}\int_0^t\left\|\nabla\boldsymbol{v}'\right\|_2\left\|\nabla\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right)\right\|_2\\
&\qquad\qquad\qquad\leq C\left(\delta,\left\|c'\right\|_{1,\infty,Q}\right)\int_0^t\left\|\boldsymbol{v}'\right\|_{1,2}^2+\delta\int_0^t\left\|\eta\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right\|_2^2.
\end{aligned}
$$

Now we can put estimates for all terms together and after we choose $\delta=\frac{\kappa}{20C_K}$ we get

$$
\begin{aligned}
&\frac{c}{2}\left\|\eta\mathrm{D}_s^h\left(\boldsymbol{v}'(t)\right)\right\|_2^2+\frac{\kappa}{2C_K}\int_0^t\left\|\eta\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right\|_2^2\leq C+\frac{C}{2}\left\|\eta\nabla\boldsymbol{v}'(0)\right\|_2^2+C\left\|\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}'\right\|_{L^2(W^{-1,2}(G))}^2\\
&\qquad+C(\kappa,\left\|c'\right\|_{1,\infty,Q})\int_0^t\left(\left\|\boldsymbol{v}'\right\|_{1,2}^2+\left\|\boldsymbol{v}'\right\|_4^4+\left\|c'\right\|_{1,4}^4+\left\|p'\right\|_2^2\right)\\
&\qquad\qquad+C(\kappa)\int_0^t\left(1+\left\|\boldsymbol{v}'\right\|_4^4+\left\|\boldsymbol{v}'\right\|_{1,2}^2\right)\left\|\eta\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right\|_2^2.
\end{aligned}\tag{3.50}
$$

Now we use Gronwall's inequality to obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]}\left\|\eta\mathrm{D}_s^h\left(\boldsymbol{v}'(t)\right)\right\|_2^2+\int_0^T\left\|\eta\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right\|_2^2\leq\\
&\quad C\left(\kappa,\left\|c'\right\|_{1,\infty,Q},\left\|\boldsymbol{v}'(0)\right\|_{1,2},\left\|\boldsymbol{v}'\right\|_{L^4(L^4)},\left\|\boldsymbol{v}'\right\|_{L^2(W^{1,2})},\left\|c'\right\|_{L^4(W^{1,4})},\left\|p'\right\|_{L^2(L^2)}\right).
\end{aligned}\tag{3.51}
$$

2. In this case we cannot use Lipschitz continuity of c', instead of that we use $\nabla c' \in L^4(0,T;L^4(G))$ and estimate the velocity using interpolation inequality in the following fashion :

$$\begin{aligned}
&\frac{1}{4}\int_0^t \int_G \mathrm{D}_s^h\left(\nu\left(c'(y)\right)\right)\tau_h^s\left(\omega_{mnkl}\right)(y)\frac{\partial \boldsymbol{v}_k'}{\partial y_l}(y)\frac{\partial}{\partial y_n}\left(\eta^2\mathrm{D}_s^h\left(\boldsymbol{v}_m'\right)\right)(y)\mathrm{dy}\\
&\qquad\leq C\int_0^t\int_G\left|\mathrm{D}_s^h\left(c'\right)\right|\left|\nabla\boldsymbol{v}'\right|\left(\left|\eta^2\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right|+\left|\eta\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right|\right)\\
&\qquad\leq C\int_0^t\|\nabla c'\|_4\|\eta\nabla\boldsymbol{v}'\|_4\left(\left\|\eta\mathrm{D}_s^h\left(\nabla\boldsymbol{v}'\right)\right\|_2+\|\nabla\boldsymbol{v}'\|_2\right)\\
&\qquad\leq C(\delta)\int_0^t\|\nabla c'\|_4^2\|\eta\nabla\boldsymbol{v}'\|_4^2+\delta\int_0^t\left\|\eta\nabla\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right\|_2^2+C(\delta)\int_0^t\|\boldsymbol{v}'\|_{1,2}^2\\
&\qquad\leq C(\delta)\int_0^t\|\nabla c'\|_4^2\|\nabla(\eta\boldsymbol{v}')\|_4^2+C(\delta)\int_0^t\|\nabla c'\|_4^4+C(\delta)\int_0^t\|\boldsymbol{v}'\|_4^4\\
&\qquad\qquad+\delta\int_0^t\left\|\eta\nabla\mathrm{D}_s^h\left(\boldsymbol{v}'\right)\right\|_2^2+C(\delta)\int_0^t\|\boldsymbol{v}'\|_{1,2}^2. \qquad (3.52)
\end{aligned}$$

Problematic part is the first term of (3.52), because instead of differences there only are derivatives of $\boldsymbol{v}'$, there is no simple way to absorb it in left hand side. From lemma 2.7 follows

$$\int_0^t\|\nabla c'\|_4^2\left\|\nabla^h(\eta\boldsymbol{v}')\right\|_4^2\to\int_0^t\|\nabla c'\|_4^2\|\nabla(\eta\boldsymbol{v}')\|_4^2 \qquad \text{for all } t\in[0,T]$$

and since functions g and g^h are nondecreasing and continuous,

$$\begin{aligned}
g(t) &= \int_0^t\|\nabla c'\|_4^2\|\nabla(\eta\boldsymbol{v}')\|_4^2\\
g^h(t) &= \int_0^t\|\nabla c'\|_4^2\left\|\nabla^h(\eta\boldsymbol{v}')\right\|_4^2
\end{aligned}$$

from lemma 2.10 follows uniform convergence of g^h to limit g. Therefore we can choose $h_0>0$, such that for all $h<h_0$

$$\int_0^t\|\nabla c'\|_4^2\left\|\nabla^h(\eta\boldsymbol{v}')\right\|_4^2\geq\int_0^t\|\nabla c'\|_4^2\|\nabla(\eta\boldsymbol{v}')\|_4^2-1 \qquad \text{for all } t\in[0,T].$$

We use this to estimate first term on the end of (3.52) and get

$$
\begin{aligned}
\int_0^t \|\nabla c'\|_4^2 \|\nabla(\eta \boldsymbol{v}')\|_4^2 &\le \int_0^t \|\nabla c'\|_4^2 \left\|\nabla^h(\eta \boldsymbol{v}')\right\|_4^2 + 1 \\
&\le 1 + C\int_0^t \|\nabla c'\|_4^4 + C\int_0^t \|\boldsymbol{v}'\|_4^4 + \int_0^t \|\nabla c'\|_4^2 \left\|\eta\nabla^h \boldsymbol{v}'\right\|_4^2 \\
&\le 1 + C\int_0^t \|\nabla c'\|_4^4 + C\int_0^t \|\boldsymbol{v}'\|_4^4 + C\int_0^t \|\nabla c'\|_4^2 \left\|\eta\nabla^h \boldsymbol{v}'\right\|_2 \left\|\eta\nabla^h \boldsymbol{v}'\right\|_{1,2} \\
&\le 1 + C\int_0^t \|\nabla c'\|_4^4 + C\int_0^t \|\boldsymbol{v}'\|_4^4 + C\int_0^t \|\boldsymbol{v}'\|_{1,2}^2 \\
&\quad + C(\delta)\int_0^t \left(1 + \|\nabla c'\|_4^4\right) \left\|\eta\nabla^h \boldsymbol{v}'\right\|_2^2 + \delta\int_0^t \left\|\eta\nabla^h(\nabla \boldsymbol{v}')\right\|_2^2 .
\end{aligned}
$$

Now we can put estimates of all terms together, put $\delta = \frac{\kappa}{20C_K}$ and sum over s.

$$
\begin{aligned}
&\frac{c}{2}\left\|\eta\nabla^h \boldsymbol{v}'(t)\right\|_2^2 + \frac{\kappa}{2C_K}\int_0^t \left\|\eta\nabla^h\nabla \boldsymbol{v}'\right\|_2^2 \\
&\le C + \frac{C}{2}\|\eta\nabla \boldsymbol{v}'(0)\|_2^2 + C\left\|\frac{\mathrm{d}}{\mathrm{dt}}\boldsymbol{v}'\right\|_{L^2(W^{-1,2}(G))}^2 + C(\kappa)\int_0^t \left(\|\boldsymbol{v}'\|_{1,2}^2 + \|\boldsymbol{v}'\|_4^4 + \|c'\|_{1,4}^4 + \|p'\|_2^2\right) \\
&\quad + C(\kappa)\int_0^t \left(1 + \|\boldsymbol{v}'\|_4^4 + \|\boldsymbol{v}'\|_{1,2}^2 + \|c'\|_{1,4}^4\right)\left\|\eta\nabla^h \boldsymbol{v}'\right\|_2^2 \qquad (3.53)
\end{aligned}
$$

Now we use Gronwall's inequality to obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left\|\eta\nabla^h \boldsymbol{v}'(t)\right\|_2^2 + \int_0^T \left\|\eta\nabla^h\nabla \boldsymbol{v}'\right\|_2^2 \le \\
&\qquad C\left(\kappa, \|\boldsymbol{v}'(0)\|_{1,2}, \|\boldsymbol{v}'\|_{L^4(L^4)}, \|\boldsymbol{v}'\|_{L^2(W^{1,2})}, \|c'\|_{L^4(W^{1,4})}, \|p'\|_{L^2(L^2)}\right). \qquad (3.54)
\end{aligned}
$$

1. & 2. Since these estimates 3.51 and 3.54 are uniform with respect to h we can use lemma 2.5 and get the same estimates for derivatives instead of for differences, but on some smaller set $(-\epsilon'', \epsilon'')\times(-\epsilon'', \epsilon'')$ for arbitrary $\epsilon'' < \epsilon'$. Since ϵ' and ϵ'' can be chosen arbitrarily close to ϵ, for G' given we can choose ϵ, ϵ', ϵ'' and η such that $G' \subset (-\epsilon'', \epsilon'') \times (-\epsilon'', \epsilon'')$ and therefore obtain the result.

□

Lemma 3.11 *The function* $\boldsymbol{v}_n'$ *defined above satisfies estimate* (3.55)*, with constant* C*, which does not depend on* n

$$
\left\|\nabla^2 \boldsymbol{v}_n'\right\|_{L^2(L^2(G'))} + \|\nabla \boldsymbol{v}_n'\|_{L^\infty(L^2(G'))} \le C \qquad (3.55)
$$

For all open $G' \subset G$*, such that* $\overline{G'} \subset G$*.*

Proof : Since c_n' is Lipschitz continuous and we assumed $\boldsymbol{v}_0$ to lie in the space $W_n^{1,2}(\Omega)$, we can use lemma 3.10 part 1 for $\boldsymbol{v}_n'$ and p_n' in order to obtain

$$\begin{aligned}&\left\|\nabla^2 \boldsymbol{v}_n'\right\|_{L^2(L^2(G'))} + \|\nabla \boldsymbol{v}_n'\|_{L^\infty(L^2(G'))} \leq \\ C &\left(\nu, \|c_n'\|_{1,\infty,Q}, \|\nabla \boldsymbol{v}'(0)\|_{2,G}, \|\nabla c_n'\|_{L^2(L^2(G))}, \|\boldsymbol{v}_n'\|_{L^4(L^4(G))}, \|p_n'\|_{L^2(L^2(G))}, \|\boldsymbol{v}_n'\|_{L^2(W^{1,2}(G))}\right).\end{aligned} \tag{3.56}$$

Now we have enough information to use lemma 3.10 part 2 and get

$$\begin{aligned}&\left\|\nabla^2 \boldsymbol{v}_n'\right\|_{L^2(L^2(G'))} + \|\nabla \boldsymbol{v}_n'\|_{L^\infty(L^2(G'))} \leq \\ &C\left(\nu, \|\boldsymbol{v}_n'(0)\|_{1,2,G}, \|\boldsymbol{v}_n'\|_{L^4(L^4(G))}, \|\boldsymbol{v}_n'\|_{L^2(W^{1,2}(G))}, \|c_n'\|_{L^4(W^{1,4}(G))}, \|p_n'\|_{L^2(L^2(G))}\right).\end{aligned} \tag{3.57}$$

Since the constant on the right hand side depends only on the quantities which are estimated in (3.42) uniformly with respect to n, we can change constant C in (3.57) to some other constant C, which does not depend on n.

□

Lemma 3.12 *Approximation of velocity $\boldsymbol{v}_n$ defined above satisfies estimate* (3.58) *with constant C, which does not depend on n*

$$\left\|\nabla^2 \boldsymbol{v}_n\right\|_{L^2(L^2(\Omega))} + \|\nabla \boldsymbol{v}_n'\|_{L^\infty(L^2(\Omega))} \leq C \tag{3.58}$$

Proof : Proof is the same as of theorem 3.9.

□

Lemma 3.13 *Let c^n be given sequence, which satisfies*

$$\begin{aligned}0 \leq c^n(x,t) \leq 1 \quad &\forall (x,t) \in Q, \\ c^n \to c \quad &\textit{in space } L^4(Q)\end{aligned}$$

and let $\boldsymbol{v}^n$, p^n be weak solution of system of equations

$$\frac{\partial \boldsymbol{v}^n}{\partial t} - \operatorname{div}\left(\nu\left(c^n\right) \mathbf{D}\left(\boldsymbol{v}^n\right)\right) + \operatorname{div}\left(\boldsymbol{v}^n \otimes \boldsymbol{v}^n\right) = -\nabla p^n + \boldsymbol{f} \tag{3.59}$$

$$\operatorname{div} \boldsymbol{v}^n = 0 \tag{3.60}$$

with initial condition $\boldsymbol{v}_0$. If also there exist constant C independent of n such that

$$\|\mathbf{D}(\boldsymbol{v}^n)\|_{L^4(L^4)} \leq C$$

than

$$\boldsymbol{v}^n \to \boldsymbol{v} \quad \textit{in space } L^\infty\left(0,T; L^2(\Omega)\right)$$

Where $\boldsymbol{v}$ is the original weak solution of the problem (2.1) - (2.3) *and therefore solves*

$$\frac{\partial \boldsymbol{v}}{\partial t} - \operatorname{div}\left(\nu\left(c\right)\mathbf{D}\left(\boldsymbol{v}\right)\right) + \operatorname{div}\left(\boldsymbol{v}\otimes\boldsymbol{v}\right) = -\nabla p + \boldsymbol{f}, \tag{3.61}$$

$$\operatorname{div}\boldsymbol{v} = 0, \tag{3.62}$$

with initial condition $\boldsymbol{v}_0$.

Proof : We define

$$\begin{aligned}\tilde{\boldsymbol{v}} &= \boldsymbol{v} - \boldsymbol{v}^n,\\ \tilde{c} &= c - c^n.\end{aligned}$$

We can use $\tilde{\boldsymbol{v}}\chi_{[0,t]}$ and $\tilde{c}\chi_{[0,t]}$ as a test function in weak formulations for both $\boldsymbol{v}$ and $\boldsymbol{v}^n$ and subtract the equations to get the equality :

$$\begin{aligned}\int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\tilde{\boldsymbol{v}}\right\|_2^2 + \int_0^t\int_\Omega \nu\left(c\right)\left|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right|^2 \\ = \int_0^t\int_\Omega \left(\nu\left(c^n\right) - \nu\left(c\right)\right)\mathbf{D}\left(\boldsymbol{v}^n\right)\mathbf{D}\left(\tilde{\boldsymbol{v}}\right) - \int_0^t\int_\Omega [\nabla\boldsymbol{v}]\tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{v}}.\end{aligned} \tag{3.63}$$

We will use Hölder's, interpolation, Young's and Korn's inequality and the fact that viscosity is Lipschitz continuous function in order to estimate right hand side of (3.63).

$$\begin{aligned}\int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\tilde{\boldsymbol{v}}\right\|_2^2 + \int_0^t\int_\Omega \nu\left(c\right)\left|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right|^2 &\le C\int_0^t\int_\Omega \left|\tilde{c}\right|\left|\mathbf{D}\left(\boldsymbol{v}^n\right)\right|\left|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right| - \int_0^t\int_\Omega [\nabla\boldsymbol{v}]\tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{v}}\\ &\le C\int_0^t \left\|\tilde{c}\right\|_4\left\|\mathbf{D}\left(\boldsymbol{v}^n\right)\right\|_4\left\|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right\|_2 + \int_0^t \left\|\nabla\boldsymbol{v}\right\|_2\left\|\tilde{\boldsymbol{v}}\right\|_4^2\\ &\le C\int_0^t \left(\left\|\tilde{c}\right\|_4\left\|\mathbf{D}\left(\boldsymbol{v}^n\right)\right\|_4\left\|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right\|_2 + \left\|\nabla\boldsymbol{v}\right\|_2\left\|\tilde{\boldsymbol{v}}\right\|_2\left\|\tilde{\boldsymbol{v}}\right\|_{1,2}\right)\\ &\le C\int_0^t \left(\epsilon\left\|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right\|_2^2 + \left\|\tilde{c}\right\|_4^2\left\|\mathbf{D}\left(\boldsymbol{v}^n\right)\right\|_4^2 + \epsilon\left\|\nabla\tilde{\boldsymbol{v}}\right\|_2^2 + \left(1 + \left\|\nabla\tilde{\boldsymbol{v}}\right\|_2^2\right)\left\|\tilde{\boldsymbol{v}}\right\|_2^2\right)\end{aligned} \tag{3.64}$$

We choose $\epsilon < \frac{\kappa}{2C_K}$ and use Korn's inequality to obtain

$$\left\|\tilde{\boldsymbol{v}}\left(t\right)\right\|_2^2 \le C\int_0^t \left(\left\|\tilde{c}\right\|_4^2\left\|\mathbf{D}\left(\boldsymbol{v}^n\right)\right\|_4^2 + \left(1 + \left\|\nabla\tilde{\boldsymbol{v}}\right\|_2^2\right)\left\|\tilde{\boldsymbol{v}}\right\|_2^2\right). \tag{3.65}$$

Since $\left\|\nabla\tilde{\boldsymbol{v}}\right\|_2^2$ and $\left\|\tilde{c}(s)\right\|_4^2\left\|\mathbf{D}\left(\boldsymbol{v}^n\right)\right\|_4^2$ are integrable, we can apply Gronwall's inequality in order to obtain

$$\begin{aligned}\left\|\boldsymbol{v}\left(t\right)\right\|_2^2 \le C\exp\left(C\int_0^t \left(1 + \left\|\nabla\tilde{\boldsymbol{v}}(s)\right\|_2^2\right)\mathrm{d}s\right)\\ \int_0^t \left\|\tilde{c}(s)\right\|_4^2\left\|\mathbf{D}\left(\boldsymbol{v}^n(s)\right)\right\|_4^2\exp\left(-C\int_0^s \left(1 + \left\|\nabla\tilde{\boldsymbol{v}}(r)\right\|_2^2\right)\mathrm{d}r\right)\mathrm{d}s\\ \le C\int_0^t \left\|\tilde{c}(s)\right\|_4^2\left\|\mathbf{D}\left(\boldsymbol{v}^n(s)\right)\right\|_4^2\mathrm{d}s.\end{aligned}$$

We take supremum over time of both sides and use Holder's inequality to estimate the right hand side and get

$$\|\tilde{\boldsymbol{v}}\|^2_{L^\infty(L^2)} \leq C \|\tilde{c}\|^2_{L^4(L^4)} \|\mathbf{D}(\boldsymbol{v}^n)\|^2_{L^4(L^4)}$$

Since $\|\mathbf{D}(\boldsymbol{v}^n)\|_{L^4(L^4)}$ is bounded and $\|\tilde{c}\|_{L^4(L^4)}$ converges to zero, we obtained the result.

□

Theorem 3.14 *Let $\boldsymbol{v}$, p and c be weak solution of system* (2.3) *in the sense of definition 2.1 and let $\nu : [0,1] \to \mathbf{R}$ be Lipschitz continuous positive function if moreover initial condition c_0 is in space $W^{1,2}(\Omega)$ and $\boldsymbol{v}_0$ is in space $W^{1,2}_n(\Omega)$, and boundary of Ω is $\mathcal{C}^4$, then*

$$\boldsymbol{v} \in L^2\left(0,T; W^{2,2}(\Omega)\right) \cap L^\infty\left(0,T; W^{1,2}(\Omega)\right)$$

Proof : Since (3.58) is uniform estimate with respect to n we can find subsequence of $\boldsymbol{v}_n$ (not relabeled) such that

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{u} \quad \text{weakly in } L^2(0,T; W^{2,2}(\Omega)) \tag{3.66}$$

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{u} \quad \text{weakly* in } L^\infty(0,T; W^{1,2}(\Omega)) \tag{3.67}$$

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{u} \quad \text{weakly* in } L^4(0,T; W^{1,4}(\Omega)) \tag{3.68}$$

Since from lemma 3.13 follows

$$\boldsymbol{v}^n \to \boldsymbol{v} \quad \text{strongly in } L^2(0,T; L^2(\Omega)) \tag{3.69}$$

it is easy to show, that $\boldsymbol{u} = \boldsymbol{v}$, which finishes the proof.

□

3.8 Uniqueness of weak solution

For two dimensional Navier - Stokes equations it is not difficult to show uniqueness of solution with no additional assumption on the solution. In case of our model we cannot use the proof of uniqueness of solution for Navier - Stokes because viscosity depends on concentration, which is part of the solution itself. This problem can be solved easily with interpolation inequality if we have higher regularity of the velocity at our disposal.

Theorem 3.15 *Let the assumptions of Theorem 2.3 be satisfied, then there exists a triple of functions $\boldsymbol{v}$, c and p, which is a weak solution of the problem* (2.1) - (2.3) *in the sense of definition 2.1 and this solution is an unique weak solution.*

Proof : Since all the assumptions of Theorems 2.2 and 2.3 are satisfied, we have that there exists weak solution $\boldsymbol{v}$, c and p to the problem (2.1) - (2.3), for which

$$\boldsymbol{v} \in L^2\left(0,T;W^{2,2}\left(\Omega\right)\right) \cap L^\infty\left(0,T;W^{1,2}\left(\Omega\right)\right). \tag{3.70}$$

Since (3.70) holds we use interpolation inequality to obtain

$$\mathbf{D}\left(\boldsymbol{v}\right) \in L^4\left(0,T;L^4\left(\Omega\right)\right). \tag{3.71}$$

Let's assume that there exists another weak solution $\boldsymbol{v}'$, c' and p' of the problem (2.1) - (2.3). We define

$$\begin{aligned} \tilde{\boldsymbol{v}} &= \boldsymbol{v} - \boldsymbol{v}', \\ \tilde{c} &= c - c'. \end{aligned}$$

We can use $\chi_{[0,t]}\tilde{\boldsymbol{v}}$ and $\chi_{[0,t]}\tilde{c}$ as a test functions in weak formulations for both solutions, and subtract equations for concentrations and balances of momentum. We start with the balance of linear momentum :

$$\begin{aligned} \int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\tilde{\boldsymbol{v}}\right\|_2^2 + \int_0^t\int_\Omega \nu\left(c'\right)\left|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right|^2 & \\ = \int_0^t\int_\Omega \left(\nu\left(c'\right)-\nu\left(c\right)\right)\mathbf{D}\left(\boldsymbol{v}\right)\cdot\mathbf{D}\left(\tilde{\boldsymbol{v}}\right) - \int_0^t\int_\Omega [\nabla\boldsymbol{v}']\tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{v}}. \end{aligned} \tag{3.72}$$

We will use Holder's, interpolation and Young's inequalities and the fact that viscosity is Lipschitz function in order to estimate right hand side of (3.72) :

$$\begin{aligned} \int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\tilde{\boldsymbol{v}}\right\|_2^2 + \int_0^t\int_\Omega \nu\left(c'\right)\left|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right|^2 &\le C\int_0^t\int_\Omega \left|\tilde{c}\right|\left|\mathbf{D}\left(\boldsymbol{v}\right)\right|\left|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right| - \int_0^t\int_\Omega [\nabla\boldsymbol{v}]\tilde{\boldsymbol{v}}\cdot\tilde{\boldsymbol{v}} \\ &\le C\int_0^t \left\|\tilde{c}\right\|_4\left\|\mathbf{D}\left(\boldsymbol{v}\right)\right\|_4\left\|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right\|_2 + \int_0^t \left\|\nabla\boldsymbol{v}\right\|_2\left\|\tilde{\boldsymbol{v}}\right\|_4^2 \\ &\le C\int_0^t \left(\left\|\tilde{c}\right\|_2^{\frac{1}{2}}\left\|\tilde{c}\right\|_{1,2}^{\frac{1}{2}}\left\|\mathbf{D}\left(\boldsymbol{v}\right)\right\|_4\left\|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right\|_2 + \left\|\nabla\boldsymbol{v}\right\|_2\left\|\tilde{\boldsymbol{v}}\right\|_2\left\|\tilde{\boldsymbol{v}}\right\|_{1,2}\right) \\ &\le \int_0^t\int_\Omega \left(\epsilon\left\|\mathbf{D}\left(\tilde{\boldsymbol{v}}\right)\right\|_2^2 + \epsilon\left\|\nabla\tilde{c}\right\|_2^2 + C\left(1+\left\|\mathbf{D}\left(\boldsymbol{v}\right)\right\|_4^4\right)\left(\left\|\tilde{\boldsymbol{v}}\right\|_2^2 + \left\|\tilde{c}\right\|_2^2\right)\right). \end{aligned} \tag{3.73}$$

We make similar estimates in equation for concentration :

$$\begin{aligned} \int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\tilde{c}\right\|_2^2 + k\int_0^t \left\|\nabla\tilde{c}\right\|_2^2 = -\int_0^t\int_\Omega \nabla c\tilde{\boldsymbol{v}}\tilde{c} &\le \int_0^t \left\|\nabla c\right\|_2\left\|\tilde{\boldsymbol{v}}\right\|_4\left\|\tilde{c}\right\|_4 \\ &\le C\int_0^t \left\|\nabla c\right\|_2\left\|\tilde{\boldsymbol{v}}\right\|_2\left\|\tilde{\boldsymbol{v}}\right\|_{1,2}\left\|\tilde{c}\right\|_2\left\|\tilde{c}\right\|_{1,2} \\ &\le C\int_0^t \left(\epsilon\left\|\nabla\tilde{\boldsymbol{v}}\right\|_2 + \epsilon\left\|\nabla\tilde{c}\right\|_2 + C\left(1+\left\|\nabla c\right\|_2^2\right)\right)\left(\left\|\tilde{\boldsymbol{v}}\right\|_2^2 + \left\|\tilde{c}\right\|_2^2\right). \end{aligned} \tag{3.74}$$

If we sum (3.73) and (3.74), choose ϵ sufficiently small and use Korn's inequality and boundedness of viscosity from bellow we obtain estimate

$$\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{dt}}\left(\|\tilde{\boldsymbol{v}}\|_2^2+\|\tilde{c}\|_2^2\right) \leq C\int_0^t \left(1+\|\nabla c\|_2^2+\|\mathbf{D}\left(\boldsymbol{v}\right)\|_4^4\right)\left(\|\tilde{\boldsymbol{v}}\|_2^2+\|\tilde{c}\|_2^2\right). \tag{3.75}$$

Since from (3.71) follows that $\left(1+\|\nabla c\|_2^2+\|\mathbf{D}\left(\boldsymbol{v}_2\right)\|_4^4\right)$ belongs to the space $L^1\left([0,T]\right)$, we can use Gronwall's inequality, from which and from the equalities $\tilde{c}(0)=0$ and $\tilde{\boldsymbol{v}}(0)=0$ follows

$$\tilde{c}=0, \qquad \tilde{\boldsymbol{v}}=0.$$

□

3.9 Possible extensions of the result

In this section we discus possible extensions of theorem 2.3 and the method being used in its proof. The method used to prove theorem 2.3 can be easily adapted to the no-slip boundary conditions it only requires to change the extension of the first component of the velocity in section 3.5 to odd instead of even extension. On the other hand I do not know about extension which is suitable for Navier partial slip boundary conditions. Generalization of the method into three spatial dimensions is probably also possible provided that we can construct the mapping T, for which Lemmas 3.1 and 3.2 hold, of course in this case we cannot handle the convective term with interpolation inequality 2.9.

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